Notes on Cartan's Method of Moving Frames

Andrejs Treibergs

The method of moving frames is a very efficient way to carry out computations on surfaces. Chern's Notes† give an elementary introduction to differential forms. However these are brief and without examples. In this handout, following Chern, I review my version of the notation and develop a couple of explicit examples. Hopefully these remarks will help you see what's going on.

Derivation of the equations. We begin by quickly reviewing Cartan's formulation of local differential geometry in terms of moving frames. Our indices shall have the range $i, j, k, \ldots = 1, 2$ and $A, B, C, \ldots = 1, 2, 3$ and we follow the convention that sums are over repeated indices. Let $S \subset \mathbf{R}^3$ be a surface and let the dot product of $\mathbf{R}^3$ be given $\langle \cdot\,, \cdot \rangle$. Let a local chart for S be given by the map $X : U \to S$ where $U \subset \mathbf{R}^2$ is an open set. In a neighborhood of a point we choose a local orthonormal frame, smooth vector fields $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ such that

$$\langle \mathbf{e}_A, \mathbf{e}_B \rangle = \delta_{AB}, \tag{1}$$

the Kronecker delta. We choose the frame adapted in such a way that $\mathbf{e}_3$ is the unit normal vector and $\mathbf{e}_1$ and $\mathbf{e}_2$ span the tangent space T_pS. The corresponding coframe field of one forms $\{\omega^A\}$ is defined by the differential

$$dX = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2. \tag{2}$$

In local coordinates $(u^1, u^2) \in U$ the one forms are linear functionals of the form

$$\omega(\cdot) = P_1(u^1, u^2)du^1 + P_2(u^1, u^2)du^2 \tag{3}$$

where P and Q are smooth functions in U. du^i are the differentials for the coordinate functions $u^i : U \to \mathbf{R}$ which form a basis for the linear functionals on the vector spaces $T_{(u^1,u^2)}U$. Thus, for vectors which in local coordinates have the expression

$$V = v^1(u^1, u^2)\frac{\partial}{\partial u^1} + v^2(u^1, u^2)\frac{\partial}{\partial u^2},$$

the one form (3) acts by

$$\omega(V) = v^1(u^1, u^2)\, P_1(u^1, u^2) + v^2(u^1, u^2)\, P_2(u^1, u^2).$$

For simplicity sake we have only described one forms on U. But by the usual identifications between $X : U \to X(U)$ or $dX : T_{(u^1,u^2)}U \to T_pS$, the one forms can be interpreted as linear functionals on T_pS as well. For example, if we choose vector fields E_i in U such that $dX(E_i) = \mathbf{e}_i$ then we may set $\tilde{\omega}(\mathbf{e}_i) := \omega(E_i)$.

†S. S. Chern, Euclidean Differential Geometry Notes (Math 140), University of California, Berkeley

In particular, $\tilde\omega^i(\mathbf{e}_A) = \delta^i{}_A$. It also means that if the metric takes the form $ds^2 = (\omega^1)^2 + (\omega^2)^2$ then ω^i is automatically a coframe and the vector fields $\mathbf{e}_i$ determined by duality $\tilde\omega^i(\mathbf{e}_j) = \delta^i{}_j$ are automatically the corresponding orthonormal frame.

One forms can be integrated on curves in the usual way. If $\alpha : [0, L] \to U$ is a piecewise smooth curve where $\alpha(t) = \big(u^1(t), u^2(t)\big)$ then

$$\int_{\alpha([0,L])} \omega := \int_0^L \omega(\alpha'(t))\, dt = \int_{\alpha([0,L])} P_1 du^1 + P_2 du^2$$

is the usual line integral. Two one forms may be multiplied (wedged) to give a two form, which is a skew symmetric bilinear form on the tangent space. In general terms, if ϑ and ω are one forms then for vector fields Y, Z we have the formula

$$(\vartheta \wedge \omega)(Y, Z) := \vartheta(Y)\omega(Z) - \vartheta(Z)\omega(Y).$$

In local coordinates this gives

$$(p_1 du^1 + p_2 du^2) \wedge (q_1 du^1 + q_2 du^2) := (p_1 q_2 - p_2 q_1) du^1 \wedge du^2.$$

Because three vectors are dependent there are no skew symmetric three forms in $\mathbf{R}^2$ and the most general two form is

$$\beta = A(u^1, u^2) du^1 \wedge du^2.$$

When evaluated on the vectors

$$V = v^1(u^1, u^2)\frac{\partial}{\partial u^1} + v^2(u^1, u^2)\frac{\partial}{\partial u^2} \qquad Z = z^1(u^1, u^2)\frac{\partial}{\partial u^1} + z^2(u^1, u^2)\frac{\partial}{\partial u^2}$$

the two form gives

$$\beta(V, Z) = A(u^1, u^2)\left(v^1(u^1, u^2) z^2(u^1, u^2) - v^2(u^1, u^2) z^1(u^1, u^2)\right).$$

A two form, say β, may be integrated over a region $R \subset U$ by the formula

$$\int_R \beta = \int_R A(u^1, u^2)\, du^1 du^2$$

where $du^1 du^2$ denotes Lebesgue measure on U.

The first fundamental form, the metric, has the expression from (2)

$$\begin{aligned} ds^2 =& \langle dX, dX\rangle \\ =& \langle \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2, \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2\rangle \\ =& (\omega^1)^2 + (\omega^2)^2. \end{aligned} \tag{4}$$

In this notation, the area form is $\omega^1 \wedge \omega^2$. That's because by using (2) to write in terms of the $\mathbf{e}_i$ basis, the area of the parallelogram spanned by $dX(\partial/\partial u)$ and $dX(\partial/\partial v)$ is

$$\omega^1\left(\frac{\partial}{\partial u}\right)\omega^2\left(\frac{\partial}{\partial v}\right) - \omega^1\left(\frac{\partial}{\partial v}\right)\omega^2\left(\frac{\partial}{\partial u}\right) = \omega^1 \wedge \omega^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right).$$

The Weingarten equations express the rotation of the frame when moved along the surface S

$$d\mathbf{e}_A = \sum \omega_A{}^B\, \mathbf{e}_B. \tag{5}$$

This equation defines the 3×3 matrix of one forms ${\omega_A}^B$ which is called the *matrix of connection forms* of $\mathbf{E}^3$. The fact that the frame is orthonormal implies that when $\delta_{AB} = \langle \mathbf{e}_A, \mathbf{e}_B \rangle$ is differentiated, using (5),

$$\begin{aligned} 0 =& d\delta_{AB} \\ =& d\langle \mathbf{e}_A, \mathbf{e}_B \rangle \\ =& \langle d\,\mathbf{e}_A, \mathbf{e}_B \rangle + \langle \mathbf{e}_A, d\,\mathbf{e}_B \rangle \\ =& \langle \sum {\omega_A}^C\, \mathbf{e}_C, \mathbf{e}_B \rangle + \langle \mathbf{e}_A, \sum {\omega_B}^C\, \mathbf{e}_C \rangle \\ =& {\omega_A}^B + {\omega_B}^A. \end{aligned} \tag{6}$$

This equation says that the matrix of connection forms is skew so there are only three distinct ${\omega_A}^B$. Geometrically it says that the motion of the frame vectors is already determined in large part by the motion of the other vectors in the frame.

The forms ${\omega_3}^i$ determine the motion of the normal vector and hence define the second fundamental form. Thus in local coordinates, the second fundamental form is given using (4),(5) and (6),

$$\begin{aligned} \mathbf{II}(\cdot,\cdot) =& -\langle d\,\mathbf{e}_3, dX \rangle \\ =& -\langle {\omega_3}^1\, \mathbf{e}_1 + {\omega_3}^2\, \mathbf{e}_2\ ,\ \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2 \rangle \\ =& -{\omega_3}^1 \otimes \omega^1 - {\omega_3}^2 \otimes \omega^2 \\ =& {\omega_1}^3 \otimes \omega^1 + {\omega_2}^3 \otimes \omega^2. \end{aligned} \tag{7}$$

We may express the connection forms using the basis

$$\begin{aligned} {\omega_1}^3 =& h_{11}\omega^1 + h_{12}\omega^2, \\ {\omega_2}^3 =& h_{21}\omega^1 + h_{22}\omega^2. \end{aligned} \tag{8}$$

Thus inserting into (7),

$$\mathbf{II}(\cdot,\cdot) = \sum h_{ij}\, \omega^i \otimes \omega^j.$$

In particular, if one searches through all unit tangent vectors

$$V_\phi := \cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2,$$

for which $\tilde{\mathbf{II}}(V_\phi, V_\phi)$ is maximum and minimum, one finds that the extrema occur as eigenvectors of h_{ij} and that the principal curvatures k_i are the corresponding eigenvalues. In particular, the Gauß and mean curvatures are

$$\begin{aligned} K &= k_1\, k_2 = \det(h_{ij}) = h_{11}h_{22} - h_{12}h_{21}, \\ H &= \tfrac{1}{2}(k_1 + k_2) = \tfrac{1}{2}\operatorname{tr}(h_{ij}) = \tfrac{1}{2}(h_{11} + h_{22}). \end{aligned} \tag{9}$$

Examples. We now work two examples showing that these forms may be found explicitly using the same amount of work that it would take using other notations. The first is the case of surfaces of revolution

$$X(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ g(u) \end{pmatrix}$$

where f and g are functions which describe the generating curve. For simplicity, let us assume

$$\begin{aligned} \dot f^2 + \dot g^2 = 1, \\ \dot f \ddot f + \dot g \ddot g = 0. \end{aligned}$$

Differentiating X we find

$$
\begin{aligned}
dX &= \begin{pmatrix} \dot f \cos v \\ \dot f \sin v \\ \dot g \end{pmatrix} du + \begin{pmatrix} -f \sin v \\ f \cos v \\ 0 \end{pmatrix} dv \\
&= du\, \mathbf{e}_1 + f dv\, \mathbf{e}_2,
\end{aligned} \tag{10}
$$

where we have taken a convenient frame

$$
\mathbf{e}_1 = \begin{pmatrix} \dot f \cos v \\ \dot f \sin v \\ \dot g \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} -\dot g \cos v \\ -\dot g \sin v \\ \dot f \end{pmatrix}.
$$

One checks that (1) holds. From (10) it also follows that

$$
\omega^1 = du, \qquad \omega^2 = f\, dv. \tag{11}
$$

Hence the first fundamental form (4) is

$$
ds^2 = du^2 + f^2\, dv^2.
$$

Now let us compute the Weingarten equations. Since there are only three independent ${\omega_A}^B$ we actually only need to work out half of this:

$$
\begin{aligned}
d\mathbf{e}_1 &= \begin{pmatrix} \ddot f \cos v \\ \ddot f \sin v \\ \ddot g \end{pmatrix} du + \begin{pmatrix} -\dot f \sin v \\ \dot f \cos v \\ 0 \end{pmatrix} dv &&= \dot f\, dv\, \mathbf{e}_2 + \frac{\ddot g}{\dot f} du\, \mathbf{e}_3 \\
d\mathbf{e}_2 &= \begin{pmatrix} -\cos v \\ -\sin v \\ 0 \end{pmatrix} dv &&= -\dot f\, dv\, \mathbf{e}_1 + \dot g\, dv\, \mathbf{e}_3 \\
d\mathbf{e}_3 &= \begin{pmatrix} -\ddot g \cos v \\ -\ddot g \sin v \\ \ddot f \end{pmatrix} du + \begin{pmatrix} \dot g \sin v \\ -\dot g \cos v \\ 0 \end{pmatrix} dv &&= -\frac{\ddot g}{\dot f} du\, \mathbf{e}_1 - \dot g\, dv\, \mathbf{e}_2,
\end{aligned}
$$

where we have used $-\ddot g \dot g = \ddot f \dot f$. It follows that the matrix of connection forms is

$$
\begin{pmatrix} {\omega_1}^1 & {\omega_1}^2 & {\omega_1}^3 \\ {\omega_2}^1 & {\omega_2}^2 & {\omega_2}^3 \\ {\omega_3}^1 & {\omega_3}^2 & {\omega_3}^3 \end{pmatrix} = \begin{pmatrix} 0 & \dot f\, dv & \frac{\ddot g}{\dot f} du \\ -\dot f\, dv & 0 & \dot g\, dv \\ -\frac{\ddot g}{\dot f} du & -\dot g\, dv & 0 \end{pmatrix}.
$$

Hence we may compute the second fundamental form (8) using (11)

$$
\begin{aligned}
{\omega_1}^3 &= \frac{\ddot g}{\dot f} du = \frac{\ddot g}{\dot f}\, \omega^1, \\
{\omega_2}^3 &= \dot g\, dv = \frac{\dot g}{f}\, \omega^2.
\end{aligned}
$$

Finally, we obtain the second fundamental form matrix

$$
\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} \frac{\ddot g}{\dot f} & 0 \\ 0 & \frac{\dot g}{f} \end{pmatrix}.
$$

Since h_{ij} is diagonal, $\mathbf{e}_i$ are the principal directions and $k_i = h_{ii}$ are the principal curvatures. Therefore,

$$K = \frac{-\ddot{f}}{f}, \qquad H = \frac{f\ddot{g} + \dot{f}\dot{g}}{2f\dot{f}}.$$

As a second example consider the Enneper's Surface

$$X(u,v) = \begin{pmatrix} u - \frac{1}{3}u^3 + uv^2 \\ v - \frac{1}{3}v^3 + vu^2 \\ u^2 - v^2 \end{pmatrix}.$$

Differentiating yields

$$dX = \begin{pmatrix} 1 - u^2 + v^2 \\ 2uv \\ 2u \end{pmatrix} du + \begin{pmatrix} 2uv \\ 1 - v^2 + u^2 \\ -2v \end{pmatrix} dv = \left(1 + u^2 + v^2\right)\left(du\,\mathbf{e}_1 + dv\,\mathbf{e}_2\right)$$

where we have taken the frame

$$\mathbf{e}_1 = \begin{pmatrix} \dfrac{1 - u^2 + v^2}{1 + u^2 + v^2} \\ \dfrac{2uv}{1 + u^2 + v^2} \\ \dfrac{2u}{1 + u^2 + v^2} \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} \dfrac{2uv}{1 + u^2 + v^2} \\ \dfrac{1 - v^2 + u^2}{1 + u^2 + v^2} \\ \dfrac{-2v}{1 + u^2 + v^2} \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} \dfrac{-2u}{1 + u^2 + v^2} \\ \dfrac{2v}{1 + u^2 + v^2} \\ \dfrac{1 - u^2 - v^2}{1 + u^2 + v^2} \end{pmatrix}.$$

Thus we have

$$\omega^1 = \left(1 + u^2 + v^2\right) du, \qquad \omega^2 = \left(1 + u^2 + v^2\right) dv.$$

It follows that

$$\begin{aligned} d\mathbf{e}_3 = {} & (1 + u^2 + v^2)\frac{\partial}{\partial u}\left(\frac{1}{1 + u^2 + v^2}\right) du\,\mathbf{e}_3 + \frac{1}{1 + u^2 + v^2}\begin{pmatrix} -2 \\ 0 \\ -2u \end{pmatrix} du \\ & + (1 + u^2 + v^2)\frac{\partial}{\partial v}\left(\frac{1}{1 + u^2 + v^2}\right) dv\,\mathbf{e}_3 + \frac{1}{1 + u^2 + v^2}\begin{pmatrix} 0 \\ 2 \\ -2v \end{pmatrix} dv. \end{aligned}$$

Hence

$$\begin{aligned} {\omega_1}^3 &= -\left\langle \mathbf{e}_1, d\mathbf{e}_3 \right\rangle = \frac{2\,du}{1 + u^2 + v^2} = \frac{2}{\left(1 + u^2 + v^2\right)^2}\omega^1, \\ {\omega_2}^3 &= -\left\langle \mathbf{e}_2, d\mathbf{e}_3 \right\rangle = \frac{-2\,dv}{1 + u^2 + v^2} = \frac{-2}{\left(1 + u^2 + v^2\right)^2}\omega^2. \end{aligned}$$

Thus

$$h_{11} = -h_{22} = \frac{2}{\left(1 + u^2 + v^2\right)^2}, \qquad h_{12} = 0$$

so $H = 0$ and $K = -4\left(1 + u^2 + v^2\right)^{-4}$.

Covariant differentiation. Covariant differentiation of a vector field Y in the direction of another vector field $V = \sum v^i \frac{\partial}{\partial u^i}$ on U is a vector field denoted $\nabla_V Y$. It is determined by orthogonal projection to the tangent space $\nabla_V Y := \operatorname{proj}(dY(V))$. Hence, in the local frame,

$$\nabla_V \mathbf{e}_i := \operatorname{proj}\left(d\mathbf{e}_i(V)\right) = \sum {\omega_i}^j(V)\mathbf{e}_j.$$

Covariant differentiation extends to all smooth vector fields V,W on U and Y, Z on S and smooth functions ϕ, ψ by the formulas

(1) $\nabla_{\phi V + \psi W} Z = \phi \nabla_V Z + \psi \nabla_W Z$ (linearity),
(2) $\nabla_V(\phi Y + \psi Z) = X(\phi)Y + \phi\nabla_V Y + X(\psi)Z + \psi\nabla_V Z$ (Leibnitz formula),
(3) $V\langle Y, Z\rangle = \langle \nabla_V Y, Z\rangle + \langle Y, \nabla_V Z\rangle$ (metric compatibility).

With these formulas one can deduce $\nabla_V\left(\sum y^i \mathbf{e}_i\right)$. As an example, we derive the formula for the geodesic curvature of a unit speed curve $\alpha(t) \in U$. Choose the frame so that say, $\mathbf{e}_2 = dX(\alpha')$. Then $\{\mathbf{e}_2, -\mathbf{e}_1\}$ is a right handed frame and the geodesic curvature is given by

$$-k_g \mathbf{e}_1 = \nabla_{\alpha'}\mathbf{e}_2 = {\omega_2}^1(\alpha')\,\mathbf{e}_1.$$

For example, consider the meridian curves of the surface of revolution in the example whose tangent vectors is $\mathbf{e}_2$. By (10) and (11) we have

$$k_g = {\omega_1}^2(\alpha') = \dot f dv\left(\frac{1}{f}\frac{\partial}{\partial v}\right) = \frac{\dot f}{f}\ .$$

Thus we recover the result that α is a geodesic if $\dot f = 0$. In a more general description we can also understand $\nabla_Z Y$ where Y, Z are vector fields on S.

Gauß equation and intrinsic geometry. We will have to differentiate (2) and (6) once more. The exterior derivative d is the differential on functions. The exterior derivative of a one form (3) is a two form given by

$$d\omega = \left(\frac{\partial P_2}{du^1}(u^1,u^2) - \frac{\partial P_1}{\partial u^2}(u^1,u^2)\right) du^1 \wedge du^2.$$

Thus $d^2 = 0$ because, for functions f,

$$\begin{aligned} d\,(df) =& d\left(\frac{\partial f}{\partial u^1}(u^1,u^2)\,du^1 + \frac{\partial f}{\partial u^2}(u^1,u^2)\,du^2\right)\\ =& \left(\frac{\partial^2 f}{\partial u^1 \partial u^2}(u^1,u^2) - \frac{\partial^2 f}{\partial u^2\partial u^1}(u^1,u^2)\right) du^1 \wedge du^2 = 0. \end{aligned}$$

The formula implies that if $f(u^1,u^2)$ is a function and ω a one form then

$$d(f\omega) = df \wedge \omega + f d\omega \qquad \text{but} \qquad d(\omega f) = d\omega f - \omega \wedge df.$$

Green's formula becomes particularly elegant:

$$\int_{\partial R} \omega = \int_{\partial R} p_1 du^1 + p_2 du^2 = \int_R \left(\frac{\partial p_2}{\partial u^1} - \frac{\partial p_1}{\partial u^2}\right) du^1 du^2 = \int_R d\omega.$$

Differentiating (2) and (5),

$$\begin{aligned} 0 = d^2 X =& d\left(\sum \omega^i \mathbf{e}_i\right) &&= \sum d\omega^i \mathbf{e}_i - \sum \omega^i \wedge d\mathbf{e}_i\\ & &&= \sum d\omega^i \mathbf{e}_i + \sum \omega^i \wedge {\omega_i}^C \mathbf{e}_C,\\ 0 = d^2 \mathbf{e}_A =& d\left(\sum {\omega_A}^B \mathbf{e}_B\right) &&= \sum d{\omega_A}^B \mathbf{e}_B - \sum {\omega_A}^B d\mathbf{e}_B\\ & &&= \sum d{\omega_A}^B \mathbf{e}_B - \sum {\omega_A}^B \wedge {\omega_B}^C \mathbf{e}_C. \end{aligned}$$

Now collect coefficients for the basis vectors $\mathbf{e}_j$ and $\mathbf{e}_C$:

$$(12)\qquad \begin{aligned} 0 &= d\omega^j - \sum \omega^i \wedge {\omega_i}^j, \\ 0 &= d{\omega_A}^C - \sum {\omega_A}^B \wedge {\omega_B}^C. \end{aligned}$$

These are called *the first and second structure equations.* By taking also the $\mathbf{e}_3$ coefficient of d^2X,

$$0 = \sum \omega^j \wedge {\omega_j}^3$$

so that by (8) and (12.2)

$$0 = \sum \omega^i \wedge \left(h_{ij}\omega^j\right) = \left(h_{12} - h_{21}\right)\omega^1 \wedge \omega^2.$$

It follows that $h_{ij} = h_{ji}$ is a symmetric matrix. In the text, we saw this when we proved the shape operator $-d\mathbf{e}_3$ was self adjoint.

The second structure equation enables us to compute the Gauß curvature from the connection matrix. Indeed, by (8),

$$(13)\qquad \begin{aligned} d{\omega_1}^2 &= \sum {\omega_1}^B \wedge {\omega_B}^2 = {\omega_1}^3 \wedge {\omega_3}^2 = -{\omega_1}^3 \wedge {\omega_2}^3 \\ &= -\left(h_{11}\omega^1 + h_{12}\omega^2\right) \wedge \left(h_{21}\omega^1 + h_{22}\omega^2\right) \\ &= -\left(h_{11}h_{22} - {h_{12}}^2\right)\omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2. \end{aligned}$$

The remarkable thing is that the conditions (6) and (12)

$$(14)\qquad \begin{aligned} d\omega^i &= \sum \omega^j \wedge {\omega_j}^i \\ {\omega_i}^j &+ {\omega_j}^i = 0 \end{aligned}$$

determine ${\omega_1}^2$ uniquely. Since ω^i is known once the metric is known by (4), this says that ${\omega_1}^2$ and thus K can be determined from the metric alone. This is *Gauß's Theorema Egregium.*

Lemma. *There is a unique one form* ${\omega_1}^2$ *which satisfies* (14) *on* U.

Proof. Try to determine the coefficients in a basis. Denote the two forms

$$d\omega^i = a_i(u^1, u^2)\omega^1 \wedge \omega^2.$$

We seek functions p_i so that ${\omega_1}^2 = -{\omega_2}^1 = p_1\omega^1 + p_2\omega^2$. Equations (14) become

$$\begin{aligned} d\omega^1 &= a_1\omega^1 \wedge \omega^2 = \omega^2 \wedge {\omega_2}^1 = \omega^2 \wedge \left(-p_1\omega^1 - p_2\omega^2\right) = p_1\,\omega^1 \wedge \omega^2 \\ d\omega^2 &= a_2\omega^1 \wedge \omega^2 = \omega^1 \wedge {\omega_1}^2 = \omega^1 \wedge \left(p_1\omega^1 + p_2\omega^2\right) \quad = p_2\,\omega^1 \wedge \omega^2. \end{aligned}$$

Thus the coefficients are uniquely determined. We set ${\omega_1}^2 = -{\omega_2}^1 = a_1\omega^1 + a_2\omega^2$ to solve the equations. □

If we are given an abstract Riemannian two manifold S determined by local coordinate charts, transition functions and a metric

$$ds^2 = g_{11}(u,v)\,du^2 + 2g_{12}(u,v)\,du\,dv + g_{22}(u,v)\,dv^2,$$

without necessarily having the embedding map $X\colon S \to \mathbf{R}^3$, the notion of an orthonormal coframe $\{\omega^i\}$ still makes sense by diagonalizing the metric (4). Hence a moving frame $\{\mathbf{e}_i\}$, wedge product $\wedge$, exterior derivative d, length and area all still make intrinsic sense. By the lemma we obtain the connection form ${\omega_1}^2$ and thus covariant differentiation and the curvature using (10), all depending only on intrinsic quantities and computations in local charts.

Computation of curvature from the metric. Let us compute the curvatures of a pair of metrics using this scheme. Here is where the efficiency of the method will become apparent.

Let us compute the curvature of a metric in orthogonal coordinates. For simplicity sake, I take the coefficients to be squares. Thus we are given the metric

$$ds^2 = E^2 du^2 + G^2 dv^2,$$

where $E(u,v), G(u,v) > 0$ are smooth functions in U. It is natural to guess that

$$\omega^1 = E\,du, \qquad \omega^2 = G\,dv.$$

Then, differentiating,

$$\begin{aligned} d\omega^1 =& E_v\,dv \wedge du = \omega^2 \wedge {\omega_2}^1 = G\,dv \wedge \frac{E_v}{G}\,du, \\ d\omega^2 =& G_u\,du \wedge dv = \omega^1 \wedge {\omega_1}^2 = E\,du \wedge \frac{Gu}{E}\,dv. \end{aligned}$$

Thus we may take

$${\omega_1}^2 = -{\omega_2}^1 = \frac{G_u}{E}\,dv - \frac{E_v}{G}\,du.$$

Hence by differentiating again,

$$-K\,\omega^1 \wedge \omega^2 = -KEG\,du \wedge dv = d{\omega_1}^2 = \left(\frac{\partial}{\partial u}\left(\frac{G_u}{E}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{G}\right)\right) du \wedge dv$$

from which it follows that

$$K = -\frac{1}{EG}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{E}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{G}\right)\right).$$

As a second example, let's do a metric which is not in orthogonal coordinates. To find the curvature of the metric

$$ds^2 = du^2 + 2u\,du\,dv + (1+v^2)dv^2$$

in the set where $1 - u^2 + v^2 > 0$ we first complete the square:

$$ds^2 = (du + u\,dv)^2 + \left(\sqrt{1-u^2+v^2}\,dv\right)^2.$$

Hence we may take as an orthonormal coframe

$$\omega^1 = du + u\,dv, \qquad \omega^2 = \sqrt{1-u^2+v^2}\,dv$$

which means that

$$\begin{aligned} d\omega^1 &= \qquad du \wedge dv \qquad = \omega^2 \wedge \left(-\frac{1}{\sqrt{1-u^2+v^2}}du\right) \\ d\omega^2 &= \frac{-u}{\sqrt{1-u^2+v^2}}\,du \wedge dv = \omega^1 \wedge \left(\frac{1}{\sqrt{1-u^2+v^2}}du\right) \end{aligned}$$

so that

$${\omega_1}^2 = \frac{1}{\sqrt{1-u^2+v^2}}\,du.$$

Thus

$$-K\omega^1 \wedge \omega^2 = -K\sqrt{1-u^2+v^2}\,du \wedge dv = d{\omega_1}^2 = -\frac{v}{(1-u^2+v^2)^{3/2}}\,du \wedge dv$$

so

$$K = \frac{v}{(1-u^2+v^2)^2}.$$