Homework for Math 6410 §1, Fall 2008

Andrejs Treibergs, Instructor

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Please read the relevant sections in the texts indicated.

1. Find the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} -10 & -6 & -3 \\ 15 & 9 & 5 \\ -3 & -2 & -2 \end{pmatrix} \mathbf{y}.$$

2. Suppose that A is a real 2×2 matrix whose eigenvalues are $a \pm ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Using the Jordan Canonical Form for complex matrices, show that A is similar to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

3. Let A be a real $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$. Consider the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{c}. \end{cases}$$

   Let $\varphi(t; \mathbf{c})$ denote its solution. Show that (Perko 20[8])

$$\lim_{\mathbf{z} \to \mathbf{c}} \varphi(t; \mathbf{z}) = \varphi(t; \mathbf{c}).$$

4. Suppose that A is a real $n \times n$ matrix and $\mathbf{c} \in \mathbb{R}^n$. Suppose that all eigenvalues of A have non-positive real part. If λ is eigenvalue such that $\Re\mathrm{e}\,\lambda = 0$ then assume that its algebraic multiplicity equals it geometric multiplicity (so the corresponding Jordan block has no "1's" on the superdiagonal.) Show that the solutions of the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(t_0) = \mathbf{c}. \end{cases}$$

   are bounded for $t \geq t_0$. (Perko 50[10].)

Do any four of the problems 5.–10. The tentative due date is Friday, Oct. 10.

5. Suppose $a > 0$ and $\alpha(t), \beta(t)$ and $u(t)$ are nonnegative continuous functions on $[-a, a]$. Assume

$$u(t) \le \alpha(t) + \left| \int_0^t \beta(s)\, u(s)\, ds \right|, \qquad \text{for all } |t| \le a.$$

Show

$$u(t) \le \alpha(t) + \left| \int_0^t \alpha(s)\, \beta(s)\, \exp\left(\left| \int_s^t \beta(\sigma)\, \sigma \right| \right) ds \right|, \qquad \text{for all } |t| \le a.$$

[Hint: consider $v(t) = \int_0^t \beta(s)\, u(s)\, ds$. From *Ordinary Differential Equations* by H. Amann.]

6. Suppose $E \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^p$ are open sets. Suppose $f(t, x, v) \in \mathcal{C}^1((a, b) \times E \times U, \mathbb{R}^n)$. Consider the non-autonomous, parameter dependent initial value problem where $(t_0, x_0, z_0) \in (a, b) \times E \times U$,

$$\begin{cases} \dfrac{\partial}{\partial t} x(t; t_0, x_0, z_0) = f\big(t, x(t; t_0, x_0, z_0), z_0\big), \\ \qquad\qquad x(t_0; t_0, x_0, z_0) = x_0. \end{cases} \tag{1}$$

Show that by extending, the new variable $X(t; t_0, x_0, z_0) = \big(y(t), s(t), z(t)\big) \in E \times (a, b) \times U$ satisfies an *autonomous* differential equation whose solution will imply the solution of (1). State a short-time existence-uniqueness theorem for the initial value problem (1) and the regular dependence on (t_0, x_0, z_0) result that you obtain using the local theory for the autonomous equation of X.

7. Suppose $a < 0 < b$, $A(t)$ is an $n \times n$ real matrix function and $b(t) \in \mathbb{R}^n$ such that $A(t)$ and $b(t)$ are continuous on $[a, b]$. Then for any $y \in \mathbb{R}^n$ there is a unique solution $x(t) \in \mathcal{C}^1([a, b], \mathbb{R}^n)$ to the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = A(t)\mathbf{x} + b(t), \\ \quad \mathbf{x}(0) = \mathbf{y}. \end{cases}$$

(We needed this fact to prove differentiability of a solution with respect to initial data. Note that previous problem does not apply here. Also you are to show that there is a global solution over the whole interval $[a, b]$.)

8. Give a proof of Peano's existence theorem using Euler polygons:

Theorem. Let $(t_0, x_0) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ where $\mathcal{U}$ is an open set and $f \in \mathcal{C}(\mathcal{U}, \mathbb{R}^n)$. Let $R = [t_0, t_0 + a] \times \overline{B_b(x_0)} \subset \mathcal{U}$, $M = \sup\{|f(t, x)| : (t, x) \in R\}$, and $\alpha = \min\{a, b/M\}$. Then there is $y \in \mathcal{C}^1([t_1, t_0 + \alpha], \overline{B_b(x_0)})$ that solves the the initial value problem

$$\dot{x} = f(t, x),$$
$$x(t_0) = x_0.$$

An Euler polygon is defined as a continuous, piecewise linear function, whose slopes are given by f at the corner points. For any $k \in \mathbf{N}$, we define equally spaced partition points $\alpha_j = t_0 + j\alpha/k$ for $j = 0 \dots k$. Then for $t \in [t_0, \alpha_1]$, let $y_k(t) = x_0 + f(t_0, x_0)(t - t_0)$. Then for $t \in [\alpha_1, \alpha_2]$ let $y_k(t) = y_k(\alpha_1) + f(\alpha_1, y_k(\alpha_1))(t - \alpha_1)$ and so on. For the j-th step, for $t \in [\alpha_j, \alpha_{j+1}]$ let $y_k(t) = y_k(\alpha_j) + f(\alpha_j, y_k(\alpha_j))(t - \alpha_j)$. Continue incrementing j until $j = k - 1$. Thus $y_k \in \mathcal{C}([t_0, t_0 + \alpha], \mathbb{R}^n)$. This gives a family of functions $\{y_k(t)\}_{k \in \mathbf{N}}$ defined on $[t_0, t_0 + \alpha]$.

9. This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

(a) Let $J = [0, 1]$ denote an interval and let $\phi \in C(J, J)$ be a continuous transformation. Show that ϕ admits at least one fixed point. (A fixed point is $y \in J$ so that $\phi(y) = y$.)

(b) Assume that $f \in C(\mathbf{R} \times [-1, 1])$ such that for some $\lambda < \infty$ and some $0 < T < \infty$ we have

$$|f(x, y_1) - f(x, y_2)| \le \lambda |y_1 - y_2|,$$
$$f(T + x, y_1) = f(x, y_1),$$
$$f(x, -1) f(x, +1) < 0$$

for all $x \in \mathbf{R}$ and all $y_1, y_2 \in [-1, 1]$. Using $\{a\}$, show that the equation $y' = f(x, y)$ has at least one solution periodic of period T.

(c) Apply (b) to $y' = a(x)y + b(x)$ where a, b are T periodic functions.

10. Prove the uniqueness theorem of Nagumo (1926).
Theorem. *Suppose $f \in C(\mathbf{R}^2)$ such that*

$$|f(t, y) - f(t, z)| \le \frac{|y - z|}{|t|}$$

for all $t, y, z \in \mathbf{R}$ such that $t \neq 0$. Then the initial value problem

$$\frac{dy}{dt} = f(t, y)$$
$$y(0) = 0$$

has a unique solution.

Show that Nagumo's theorem implies the uniqueness statement in the Picard-Lindelöf Theorem.

Do any five of the problems 11.–20. The tentative due date is Monday, Nov. 3.

11. Consider the differential equation where a and b are positive parameters

$$\dot{x} = -\frac{ax}{\sqrt{x^2 + y^2}}$$
$$\dot{y} = -\frac{ay}{\sqrt{x^2 + y^2}} + b$$

which models the flight of a projectile heading toward the origin, that is moved off course by a constant force with strength b. Determine the conditions on a and b to ensure that a solution starting at $(p, 0)$, for $p > 0$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, p. 86.]

12. Suppose that γ is a periodic orbit of the flow $\dot{x} = f(x)$ on $\mathbb{R}^2$ where $f \in \mathcal{C}^1(\mathbb{R}^2)$. Prove that γ surrounds a rest point, that is, the bounded component of $\mathbb{R}^2 - \gamma$ contains a point where f vanishes. [Chicone, p. 88.]

13. Prove that the ω-limit set of an orbit of a gradient system consists entirely of rest points. [Chicone, p. 88.]

14. Prove the following theorem.
Theorem. Let $X \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(X, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(X, \mathbb{R})$. Show that if $\operatorname{div}(\rho f) \neq 0$ for all of X then the equation $x' = f(x)$ has at most one periodic solution.

Use this to show that the van der Pol oscillator

$$\dot{x} = y$$
$$\dot{y} = -x + \lambda(1 - x^2)y$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, p. 90.]

15. Show that the system (2) has exactly one nontrivial periodic solution. (Hint: use the Poincaré - Bendixon Theorem and the previous problem.) [*cf.* Amann, *Ordinry Differenial Equations,* p. 349.]

$$\begin{aligned} \dot{x} &= -y - x + (x^2 + 2y^2)x \\ \dot{y} &= x - y + (x^2 + 2y^2)y \end{aligned} \tag{2}$$

16. Determine the ω-limit set of the solution of the system

$$\dot{x} = 1 - x + y^3$$
$$\dot{y} = y(1 - x + y)$$

with initial condition $x(0) = 10$, $y(0) = 0$. [Chicone, p. 92.]

17. Let $T \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$. Consider the difference equation

$$\begin{aligned} x(0) &= x, \\ x(n+1) &= T(x(n)). \end{aligned} \tag{3}$$

Writing $Tx := T(x)$, a solution sequence of (3) can be given as the n-th iterates $x(n) = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. A set $H \subset \mathbb{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ $(H \subset T(H))$. H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. A closed invariant set is *invariantly connected* if it is not the union of two nonempty disjoint invariant closed sets. A solution $T^n x$ is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (3). $T_n x$ (defined for all $n \in \mathbf{Z}$) is called an *extension of the solution* $T^n x$ *to* $\mathbf{Z}$ if $T_0 x = x$ and $T(T_n x) = T_{n+1} x$ for all $n \in \mathbf{Z}$. Thus $T_n x = T^n x$ for $n \geq 0$.

(a) Show that a finite set (a finite number of points) is invariantly connected if and only if it is a *periodic motion.* [Hint: Any permutation can be written as a product of disjoint cycles.]

(b) Show that a set H is invariant if and only if each motion starting in H has an extension to $\mathbf{Z}$ that is in H for all n.

(c) Show, however, that an invariant set H may have an extension to $\mathbf{Z}$ from a point in H which is not in H.

[LaSalle in Hale's MAA Studies in ODE, p.7]

18. The topological classification of flows can be extended to nonhyperbolic flows at least in the plane.
Theorem. *If $A \in \mathbb{M}_{2\times 2}(\mathbf{R})$ has at least one eigenvalue with zero real part, then the planar system $y' = Ay$ is topologiclly equivalent to precisely one of the five following linear systems with the indicated coefficient matrices:*

(a) $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$*: the zero matrix;*

(b) $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$: *one negative and one zero eigenvalue;*

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: *one positive and one zero eigenvalue;*

(d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: *two zero eigenvalues but one eigenvector;*

(e) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: *two purely imaginary eigenvalues;*

- Prove that if A has two purely imaginary eigenvalues then the flow of the system is topologically equivalent to the flow given by (e).
- Show that the five matrices (a)–(e) generate topologically nonequivalent flows. [Hale & Koçak p.246]

19. Use successive approximations to find the solutions $x(t,a)$ and the stable manifold in the system. ($P_s x(0,a) = a$ and $x(t,a) \to 0$ as $t \to \infty$.)

$$\begin{aligned} \dot{x}_1 &= -\, x_1 \\ \dot{x}_2 &= \ \ x_2 + x_1^2. \end{aligned}$$

Check by solving the system to find the stable and unstable manifolds. [Hint. The system can be written $x' = Ax + f(x)$. Let $x^{(0)}(t,a) = 0$ and use the method of succesive approximations $x^{(n+1)}(t,a) = \mathcal{N} x^{(n)}(t,a)$ to look for a solution of the integral equation $x = \mathcal{N} x$. Let $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^\ell = E_s \oplus E_u$ be the decomposition into stable and unstable spaces, $a \in E_s$ with $|a|$ small, $P_s : E \to E_s$ be the orthogonal projection to the stable space and $f_s = P_s f$. (Similar for P_u.) [Perko, 2.7, # 2]

$$\mathcal{N} x = e^{tA} P_s x(0) + \int_0^t e^{(t-s)A} f_s(x(s))\, ds - \int_t^\infty e^{(t-s)A} f_u(x(s))\, ds.$$

20. Consider the systems

$$\begin{cases} \dot{y}_1 = -\, y_1 \\ \dot{y}_2 = -\, y_2 + y_3^2 \\ \dot{y}_3 = \ \ y_3 \end{cases} \qquad \begin{cases} \dot{z}_1 = -\, z_1 \\ \dot{z}_2 = -\, z_2 \\ \dot{z}_3 = \ \ z_3 \end{cases}$$

Let $y(t) = \phi(t, y_0)$ and $z(t) = e^{tA} z_0$ denote the solutions. Find a local homeomorphism H so that near zero, $H \circ \phi(t, x_0) = e^{tA} H(x_0)$. Use the homeomorphism to find the stable and unstable manifolds $\mathcal{W}^s(0) = H^{-1}(E_s)$ and $\mathcal{W}^u(0) = H^{-1}(E_u)$. Hint: use successive approximations to find the homeomorphism corresponding to the time-one map. Then recover homeomorphism for the flow. Answer:

$$H(y_1, y_2, y_3) = (y_1, y_2 - \tfrac{1}{3} {y_3}^2, y_3)^T,$$
$$\mathcal{W}^s(0) = \{y \in \mathbf{R}^3 : y_3 = 0\},$$
$$\mathcal{W}^u(0) = \{y \in \mathbf{R}^3 : y_1 = 0,\ y_2 = \tfrac{1}{3} {y_3}^2\}.$$

[Perko 2.8, # 1.]