Homework for Math 6410 §1, Fall 2009

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Please read the relevant sections in the texts indicated. Each problem is due six class days after its assignment, or on Dec. 17, whichever comes first.

1. [Aug. 24.] **Compute a Phase Portrait using a Computer Algebra System.** This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

   `http://www.math.utah.edu/~treiberg/M6412eg1.mws`
   `http://www.math.utah.edu/~treiberg/M6412eg1.pdf`

   or my lab notes from Math 2280,

   `http://www.math.utah.edu/~treiberg/M2282L4.mws`.

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. Explain why your system satisfies this. (Everyone in class should have a different ODE.) Using your favorite computer algebra system, *e.g.,* MAPLE or MATLAB, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include trajectories that indicate the stable and unstable directions at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any seperatrices.

2. [Aug. 26.] **Stability of Fixed Points.** The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

$$\dot{S} = \mu K - \beta SI - \mu S$$
$$\dot{I} = \beta SI - \gamma I - \mu I - \alpha I$$
$$\dot{R} = \gamma I - \mu R$$

   (a) Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

(b) Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.

(c) Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

[From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

3. [Aug. 28.] **The Contraction Mapping Principle.** Here is the abstract idea behind the Picard Theorem. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach Space (a complete normed linear space). Let $0 < b < \infty$ and $0 < k < 1$ be constants and let $T : \mathcal{X} \to \mathcal{X}$ be a transformation. Suppose that for any $\phi, \psi \in \mathcal{X}$ if $\|\psi\| \le b$ then $\|T(\psi)\| \le b$ and if $\|\phi\| \le b$ and $\|\psi\| \le b$ then

$$\|T(\psi) - T(\phi)\| \le k\|\psi - \phi\|,$$

i.e., T is a *cantraction.* Prove that there exists an element ϕ with $\|\phi\| \le b$ such that $\phi = T\phi$, that is, T has a fixed point. Prove that ϕ is the unique fixed point among points that staisfy $\|\phi\| \le b$. [Coddington & Levinson, *Theory of Ordinary Differential Equations,* Krieger 1984, pp. 40-41.]

4. [Aug. 31.] **Existence Using the Contraction Mapping Principle.** Let $a > 0$ and $b > 0$ be constants and $\mathcal{R} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \ |t| < a \text{ and } |x| < b\ \}$. Suppose $f(t, x) \in \mathcal{C}(\mathcal{R}, \mathbb{R}^n)$ satisfies a Lipschitz condition in x. That is, there is an $M < \infty$ such that for every pair $(t, x), (t, y) \in \mathcal{R}$,

$$|f(t, x) - f(t, y)| \le M|x - y|.$$

Show that for sufficiently small $r > 0$, the operator

$$T(\phi)(t) = \int_0^t f(s, \phi(s))\, ds, \qquad \text{for } |t| \le r$$

satisfies the conditions of The Contraction Mapping Principle (Aug. 28 HW) for the Banach Space $\mathcal{X} = \mathcal{C}([-r, r], \mathbb{R}^n)$ with norm $\|\psi\| = \max\{|\psi(t)| : |t| \le r\}$. Use it to show that the initial value problem

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \\ x(0) &= 0, \end{aligned}$$

has a unique solution on $|t| \le r$. [*ibid.*]

5. [Sept. 2.] **Nagumo's Uniqueness Theorem.** Prove the uniqueness theorem of Nagumo (1926).

Theorem. Suppose $f \in C(\mathbb{R}^2)$ such that

$$|f(t, y) - f(t, z)| \le \frac{|y - z|}{|t|}$$

for all $t, y, z \in \mathbb{R}$ such that $t \ne 0$. Then the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(0) = 0 \end{cases}$$

has a unique solution.

Show that Nagumo's theorem implies the uniqueness statement in the Picard Theorem.

6. [Sept. 4.] **Solve a Delay-Differential Equation.** The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \le 0$. Apply Picard's Iteration to show the local existence of a solution to the delay differential equation.

   Theorem. Let $b > 0$. Let $f \in \mathcal{C}(\mathbb{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$,

   $$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|).$$

   Let $g \in \mathcal{C}(\mathbb{R})$ such that $g(t) \le t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbb{R}\big)$ such that $|\varphi(t) - \varphi(0)| \le b$ for all $t \le 0$. Show that there is an $r > 0$ such that the initial value problem

   $$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) \\ x(t) = \varphi(t) \qquad \text{for all } t \le 0. \end{cases}$$

   has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbb{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbb{R}\big)$.

   [*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

7. [Sept. 9.] **Find a Periodic Solution.** This exercise gives conditions for an ordinary differential equation to admit periodic solutions.

   (a) Let $J = [0, 1]$ denote an interval and let $\phi \in C(J, J)$ be a continuous transformation. Show that ϕ admits at least one fixed point. (A fixed point is $y \in J$ so that $\phi(y) = y$.)

   (b) Assume that $f \in C(\mathbb{R} \times [-1, 1])$ such that for some $\lambda < \infty$ and some $0 < T < \infty$ we have

   $$\begin{gathered} |f(t, y_1) - f(x, y_2)| \le \lambda |y_1 - y_2|, \\ f(T + t, y_1) = f(t, y_1), \\ f(t, -1) f(t, +1) < 0 \end{gathered}$$

   for all $t \in \mathbb{R}$ and all $y_1, y_2 \in [-1, 1]$. Using {a}, show that the equation $y' = f(t, y)$ has at least one solution periodic of period T.

   (c) Apply (b) to $y' = a(t)y + b(t)$ where a, b are T periodic functions.

8. [Sept. 11.] **Escape Times.** Show that each solution $\big(x(t), y(t)\big)$ of the initial value problem

   $$\begin{cases} x' = & y + x^2 \\ y' = & x + y^2 \end{cases} \qquad \begin{cases} x(0) = & x_0 \\ y(0) = & y_0 \end{cases}$$

   with $x_0 > 0$ and $y_0 > 0$ cannot exist on an interval of the form $[0, \infty)$.

   [*cf.* Wilson, Ordinary Differential Equations, Addison-Wesley, 1971, p.255.]

9. [Sept. 14.] **Variation of Parameters Formula.** Solve the inhomogeneous linear system

   $$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

   where

   $$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

10. [Sept. 16.] **Jordan Form.** Find the generalized eigenvectors, the Jordan form and the general solution

$$\dot{\mathbf{y}} = \begin{pmatrix} 6 & 6 & 4 \\ -2 & -2 & -4 \\ 2 & 6 & 8 \end{pmatrix} \mathbf{y}.$$

11. [Sept. 18] **Application of Liouville's Theorem.** Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases} x'' + \dfrac{1}{t}\,x' + x = 0 \\ x(0) = 1, \qquad x'(0) = 0 \end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t \to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

12. [Sept. 21] **Trouble Lurks Near Every Point in a Linear System.** Suppose at least one eigenvalue of the real $n \times n$ matrix A has a positive real part. Prove that for any $v \in \mathbb{R}^n$, $\varepsilon > 0$ there is a solution to $x' = Ax$ so that

$$|x(0) - v| < \varepsilon \quad \text{and} \quad \lim_{t\to\infty} |x(t)| = \infty.$$

[*cf.* Hirsch & Smale, Differential Equations, Dynamical Systems and Linear Algebra, Academic Press, 1974, p. 137.]

13. [Sept. 23.] **To Use Jordan Form or Not to Use Jordan Form.** Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations.

    (a) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a matrix B with distinct eigenvalues so that $\|A - B\| \le \epsilon$.

    (b) Give two proofs of $\det(e^A) = e^{\operatorname{trace}(A)}$.

    (c) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}. \tag{1}$$

Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.,* Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

(d) Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (1) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

(e) [Optional.] Find all continuous scalar valued functions $f \in C(\mathcal{M}_{n\times n}(\mathbb{C}), \mathbb{C})$ so that

$$f(AB) = f(A)f(B) \qquad \text{for all } A, B.$$

You can probably find several different arguments on your own. [*ibid.*; or Kurosh, *Higher Algebra*, p. 334.]

14. [Sept. 25.] **. . . Trouble, Right Here in River City. . .** Let $\phi(t)$ be real, continuous and periodic with period π. Consider the scalar equation

$$y''(t) - (\cos^2 t)\, y'(t) + \phi(t)\, y(t) = 0, \qquad t \in \mathbb{R}.$$

Show that there is a solution that goes to ∞ as $t \to \infty$. [*cf.* James H. Liu, A First Course in the Qualitative Theory of Differential Equations, Prentice Hall 2003, p. 162.]

15. [Sept 28.] **. . . With a Capital "T" that Rhymes with "P". . .** Consider the T-periodic non-autonomous linear differential equation

$$\dot{x} = A(t)\, x, \qquad x \in \mathbb{R}^n, \qquad A(t+T) = A(t).$$

Let $\Phi(t)$ be the fundamental matrix with $\Phi(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t+T) = \mu\chi(t)$, where μ is an eigenvalue of $\Phi(T)$.

(b) Suppose that $\Phi(T)$ has n distinct eigenvalues μ_i, $i = 1, \ldots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\rho_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to μ_i?

(c) Consider the equation $\dot{x} = f(t)\, A_0\, x$, $x \in \mathbb{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

16. [Sept. 30.] **Discrete Dynamical Systems.** Let $T \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$. Consider the difference equation

$$\begin{aligned} x(0) &= x, \\ x(n+1) &= T(x(n)). \end{aligned} \tag{2}$$

Writing $Tx := T(x)$, a solution sequence of (2) can be given as the n-th iterates $x(n) = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. The *forward orbit* of a point x is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. A set $H \subset \mathbb{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ ($H \subset T(H)$). H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. A closed invariant set is *invariantly connected* if it is not the union of two nonempty disjoint invariant closed sets. The solution $T^n x$ starting from a given point x is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (2). $T_n x$ (defined for all $n \in \mathbf{Z}$) is called an *extension of the solution* $T^n x$ *to* $\mathbf{Z}$ if $T_0 x = x$ and $T(T_n x) = T_{n+1} x$ for all $n \in \mathbf{Z}$. Thus $T_n x = T^n x$ for $n \geq 0$.

(a) Show that a finite set (a finite number of points) is invariantly connected if and only if it is a periodic orbit. [Hint: Any permutation can be written as a product of disjoint cycles.]

(b) Show that a set H is invariant if and only if each motion starting in H has an extension to $\mathbf{Z}$ that is in H for all n.

(c) Show, however, that an invariant set H may have an extension to $\mathbf{Z}$ from a point in H which is not in H.

[J. P. LaSalle in J. Hale's *Studies in ODE*, Mathematical Association of America, 1977, p. 7]

17. [Oct. 2.] **Polar Coordinates.** Consider the differential equation where a and b are positive parameters

$$\dot{x} = -\frac{ax}{\sqrt{x^2+y^2}}$$
$$\dot{y} = -\frac{ay}{\sqrt{x^2+y^2}} + b$$

which models the flight of a bird heading toward the origin at constant speed, that is moved off course by a steady wind of velocity b. Determine the conditions on a and b to ensure that a solution starting at $(p, 0)$, for $p > 0$ reaches the origin. Hint: change to polar coordinates and study the phase portrait of the differential equation on the cylinder. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 86.]

18. [Oct. 5.] **Stationary Points of a Hamiltonian System.** Show that the system is Hamiltonian.

$$\dot{x} = (x^2 - 1)(3y^2 - 1)$$
$$\dot{y} = -2xy(y^2 - 1)$$

Find the equilibrium points and classify them. Find the Hamiltonian. Using obvious exact solutions and the Hamiltonian property, draw a rough sketch of the phase diagram. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 79.]

19. [Oct. 7.] **Asymptotic Stability.** Consider the differential equation on $(t, x) \in [0, \infty) \times \mathbb{R}^n$

$$x'(t) = A\,x(t) + B(t)\,x(t)$$

where all eigenvalues of the constant matrix A have negative real parts and $B(t)$ is continuous and $B(t) \to 0$ as $t \to \infty$. Show that the zero solution is uniformly asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations*, Prentice Hall 2003, p. 242.]

20. [Oct. 9.] **Feedback Control.** Consider the equation for the pendulum of length ℓ, mass m in a viscous medium with friction proportional to the velocity of the pendulum. Suppose that the objective is to stabilize the pendulum in the vertical position (above its pivot) by a control mechanism which can move the pendulum horizontally. Let us assume that ϑ is the angle from the vertical position measured in a clockwise direction and the restoring force v due to the control mechanism is a linear function of ϑ and $\dot{\vartheta}$, that is, $v(\vartheta, \dot{\vartheta}) = c_1\vartheta + c_2\dot{\vartheta}$. Explain why the differential equation describing the motion is

$$m\ddot{\vartheta} + k\dot{\vartheta} - \frac{mg}{\ell}\,\sin\vartheta - \frac{1}{\ell}(c_1\vartheta + c_2\dot{\vartheta})\,\cos\vartheta = 0.$$

Show that constants c_1 and c_2 can be chosen in such a way as to make the equilibrium point $(\vartheta, \dot{\vartheta}) = (0, 0)$ asymptotically stable. [*cf.* J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 277.]

21. [Oct. 19.] **Stability of a Periodic Orbit.** Find a periodic solution to the system

$$\begin{aligned}\dot{x} &= x - y - x(x^2+y^2)\\ \dot{y} &= x + y - y(x^2+y^2)\\ \dot{z} &= -z,\end{aligned}$$

and determine its stability type. In particular, compute the Floquet Multipliers for the fundamental matrix associated with the periodic orbit. Is it orbitally asymptotically stable? Is it asymptotically stable? [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 196.]

22. [Oct. 21.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

    (a) Let A be a complex $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\|\cdot\|$ on $\mathbb{C}^n$ so that $\|Ax\| \le \gamma\|x\|$ for all $x \in \mathbb{C}^n$.

    (b) Let $P \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbb{R}^n$

$$\begin{aligned}x_{n+1} &= P(x_n),\\ x_1 &= \xi.\end{aligned}$$

23. [Oct. 23.] **Liapunov Function.** Show that the zero solution is asymptotically stable

$$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$

Hint: Show that this equation is equivalent to the system

$$\begin{aligned}\dot{x} &= y - x^3\\ \dot{y} &= -x + 2x^3 - 2y.\end{aligned}$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

24. [Oct. 26.] **Dulac's Criterion.** Prove the following theorem.
Theorem. Let $X \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(X, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(X, \mathbb{R})$. Show that if $\operatorname{div}(\rho f) \neq 0$ for all of X then the equation $x' = f(x)$ has at most one periodic solution in X.

Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \neq 0$)

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + \lambda(1 - x^2)y\end{aligned}$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2+y^2-1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

25. [Oct. 28.] **International Whaling Commission Model.** A simple rescaled delay difference equation for modeling the population u_n of sexually mature baleen whales is

$$u_{n+1} = su_n + R(u_{n-T}), \qquad 0 < s < 1,$$

where T is an integer corresponding to time to sexual maturity and R is the number that augments the adult population from births T years earlier. If

$$R(u) = (1-s)[1 + q(1-u)]u$$

where $q > 0$ describes fecundity increase due to low density and the delay is $T = 1$, derive the condition for a positive steady state u^* to be stable and find for which q it holds. [J. D. Murray, *Mathematical Biology,* Biomathematics Texts 19, Springer 1989, p. 62.]

26. [Oct. 30.] **Existence of a Periodic Orbit.** A model for an autocatalytic chemical reaction is given by the nondimensionalized Brusselator System

$$\begin{aligned} \dot{x} &= 1 - 4x + x^2 y, \\ \dot{y} &= 3x - x^2 y; \end{aligned}$$

where $x, y \geq 0$ correspond to concentrations. Show that the trapezoidal region with vertices $(\frac{1}{4}, 0)$, $(13, 0)$, $(1, 12)$, $(\frac{1}{4}, 12)$ is a forward invariant set for this system. Show that it has a nonconstant periodic trajectory. [University of Utah Preliminary Examination in Differential Equations, Autumn 2004.]

27. [Nov. 2.] **Periodic Orbit in Predator-Prey System.** A generalization of a predator-prey system given by Brauer and Castillo-Chavez is

$$\begin{aligned} \dot{x} &= x\left(1 - \frac{x}{30} - \frac{y}{x+10}\right), \\ \dot{y} &= y\left(\frac{x}{x+10} - \frac{1}{3}\right). \end{aligned}$$

   (a) Show that the fixed points are $(0, 0)$, $(30, 0)$ and $(5, 12.5)$ and have saddle, saddle, source type, resp.
   (b) Show that the region bounded by $0 \leq x$, $0 \leq y$ and $x + y \leq 50$ is forward invariant.
   (c) Show that there is no orbit Γ with $\alpha(\Gamma) = \{(30, 0)\}$ and $\omega(\Gamma) = \{(0, 0)\}$. Conclude that there is a nonconstant periodic orbit.

[R. C. Robinson, *An Introduction to Dynamical Systems Continuous and Discrete*, Pearson/Prentice Hall, 2004, p. 238.]

28. [Nov. 4.] **Stable and Unstable Manifolds.** Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\begin{aligned} \dot{\mathbf{x}} &= -x \\ \dot{y} &= -y + x^2 \\ \dot{z} &= z + y^2. \end{aligned}$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 116–117.]

29. [Nov. 6.] **Center Manifold.** Find a center manifold for the system

$$\begin{aligned} \dot{x} &= -xy \\ \dot{y} &= -y + x^2 - 2y^2 \end{aligned}$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_x x^k$$

using the condition for invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi'(0) = \psi(0) = 0$. Find the first few terms of the expansion, guess the answer and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

30. [Nov. 9.] **Stability at a Non-hyperbolic Critical Point.** Show that the origin is asymptotically stable for the system

$$\begin{aligned}\dot{x} &= -y + yz + (y - x)(x^2 + y^2),\\ \dot{y} &= x - xz - (x + y)(x^2 + y^2),\\ \dot{z} &= -z + (1 - 2z)(x^2 + y^2).\end{aligned}$$

Hint: Show that the surface $z = x^2 + y^2$ is invariant. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 435.]

31. [Nov. 11.] **Hartman-Grobman Theorem.** Find a homeomorphism H in a neighborhood of 0 that establishes an isochronous flow equivalence between the flow of the differential system and the flow of the linearized system, *i.e.,* $H(\phi(t, x)) = e^{tA}H(x)$ where $A = Df(0)$ and $\phi(t, x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$, the nonlinear system given by

$$\begin{aligned}\dot{x} &= -x\\ \dot{y} &= -y + xz,\\ \dot{z} &= z.\end{aligned}$$

[In 8.5.10, Liu discusses the approximation used in the proof, but you can guess H from the solutions and verify.]

32. [Nov. 13.] **Bifurcation in a Forest Model.** Consider Ludwig's model for the dynamics of a balsam fir forest infested by the spruce budworm. The condition of the forest is described by $S(t)$, the average size of trees and $E(t)$, the "energy reserve," a measure of the forests health. In the presence of a constant budworm population B, the forest dynamics is given by

$$\begin{aligned}\dot{S} &= r_S S\left(1 - \frac{S}{K_S}\frac{K_E}{E}\right),\\ \dot{E} &= r_E E\left(1 - \frac{E}{K_E}\right) - P\frac{B}{S},\end{aligned}$$

where r_S, r_E, K_S, K_E, P are positive parameters. Nondimensionalize the system. Sketch the nullclines. Show that there are two fixed points if B is small and none if B is large. Analyze the bifurcation at the critical value of B. What kind of bifurcation is it and why? Sketch the phase portraits for both large and small B. [S. Strogatz, *Nonlinear Dynamics and Chaos*, Westview 1994, p. 285.]

33. [Nov. 16.] **Continuation from Harmonic Oscillator.** Show that Rayleigh's Equation has a periodic solution for small ε parameter values that is a continuations from an $\varepsilon = 0$ solution

$$\ddot{x} + \varepsilon\left(\dot{x} - \dot{x}^3\right) + x = 0.$$

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, pp. 318–324.]

34. [Nov. 18.] **Bifurcation Equation.** Find and classify the bifurcations that occur as μ varies. Find the bifurcation equation and use it to determine the stability of the rest points.

$$\begin{aligned}\dot{x} &= y - 2x,\\ \dot{y} &= \mu - y + x^2,\end{aligned}$$

[S. Strogatz, *Nonlinear Dynamics and Chaos*, Westview 1994, p. 285.]

35. [Nov. 20.] **Period Doubling Bifurcations in a Discrete Dynamical System.** For discrete dynamical systems, the fixed points bifurcate when an eigenvalue of the linearization crosses the unit circle. Crossing at $\rho = -1$ may result in a period doubling bifurcation. Show that the difference equation undergoes a period doubling bifurcation at $\lambda = 1$. Can you determine the stability type of the resulting period two orbit?

$$x_{n+1} = -\lambda \arctan x_n$$

[See Liu, §7.3 and J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 92.]

36. [Nov. 23.] **Poincaré-Andronov-Hopf Bifurcation.** Show that the system undergoes a Hopf Bifurcation as the parameter passes through 0.

$$\ddot{x} + (x^2 + \dot{x}^2 - \lambda)\dot{x} + x = 0.$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 421.]

37. [Nov. 25.] **Bifurcation in the Brusselator.** Show that the system undergoes a supercritical Hopf Bifurcation as the parameter passes through 2.

$$\dot{x} = 1 - (1 + \lambda)x + x^2 y.$$
$$\dot{y} = \lambda x - x^2 y.$$

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 103.]

38. [Nov. 30.] **Feedback Stiffness Control.** Moon and Rand [1985] proposed a method of damping low modes in the vibration of trusses x by actively tensioning stiffening cables v.

$$\dot{x} = y,$$
$$\dot{y} = -x - xv,$$
$$\dot{v} = -v + \alpha x^2.$$

Show that the origin $(x, y, v) = (0, 0, 0)$ is asymtotically stable if $\alpha < 0$ and unstable if $\alpha > 0$. [Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 188, `http://audiophile.tam.cornell.edu/randpdf/moon.pdf`]

39. [Dec. 2.] **Poincaré Normal Form.** Write the system in terms of the complex variable $z = x + iy$.

$$\dot{x} = -y - xy + 2y^2,$$
$$\dot{y} = x - x^2 y.$$

By making a near-identity change of variables, put the system in Poincaré Normal Form

$$\dot{w} = \lambda w + \ell_1 w^2 \bar{w} + \mathbf{O}(|w|^4)$$

near $w = 0$. Is the origin stable?

[Y. Kuznetsov, *Elements of Applied Bifurcation Theory, 3rd. ed.*, Springer, 2004, p. 103.]

40. [Dec. 4.] **Stable and Unstable Manifold of a Poincaré Map.** Suppose we seek 2π-periodic solutions of the equation with periodic forcing

$$\ddot{x} + 2\dot{x} - 2x = 10 \cos t, \qquad \dot{x} = y.$$

Write the equation as a first order autonomous system in $(x, y, t) \in \mathbb{R}^2 \times \mathbb{S}^1$, where t is regarded as the angle in the circle $\mathbb{S}^1$. The $t = 0$ plane is a Poincaré secton for the particular solution. The Poincaré map is then defined as the flow from $t = 0$ to $t = 2\pi$, and its fixed points are 2π periodic solutions of the equation. Find the Poincaré map defined by $(P(x, y), 2\pi) = \phi(2\pi, (x, y, 0))$. Then find its stable and unstable manifolds. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 495.]

41. [Dec. 7.] **Lagrange Standard Form.** Find approximately the amplitude of the limit cycle and its period, and the polar equations for the phase paths near the limit cycle.

$$\ddot{x} + \varepsilon(|x| - 1)\dot{x} + x = 0.$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 143.]

42. [Dec. 9.] **Averaging.** Use the averaging method on

$$\ddot{x} + \varepsilon \sin(\dot{x}) + x = 0, \qquad 0 < \varepsilon \ll 1.$$

and show that the amplitude satisfies approximately

$$\dot{a} = \varepsilon J_1(a).$$

Use the formula for the first Bessel function

$$J_1(a) = \frac{1}{\pi} \int_0^\pi \sin(a \sin u) \, \sin u \, du$$

Also find the approximate differential equation for θ. Using the graph of $J_1(a)$, decide how many limit cycles the system has. Which are stable? [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 147.]

43. [Dec. 11.] **Periodic Solutions from Averaging.** Find the Lagrange Standard Form and the averaged equations. Find periodic solutions predicted by the averaged equations and determine their stability type.

$$\dot{x} = y + \varepsilon(x^2 - 1) \sin 2t.$$
$$\dot{y} = -4x.$$

[F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems, 2nd ed.,* Springer, 2006, p. 164.]