The Hartman-Grobman Theorem

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The URL for these Beamer Slides: *"Hartman-Grobman Theorem"*

`http://www.math.utah.edu/~treiberg/M6414HartmanGrobman.pdf`

This talk "Hartman-Grobman Theorem" was originally presented on overhead slides to my Math 6210-1 in 2002. These Beamer slides are developed from those slides.

- Carmen Chicone, *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics **34**, Springer, New York, 1999.
- D. Grobman, Homeomorphisms of systems of differential equations, (Russian), Dokl. Akad., Nauk. **128**, (1959) 880–881.
- Philip Hartman, A Lemma in the theory of structural stability of differential equations, Proc. Amer. Math. Soc., **11** (1960) 610–620.
- Philip Hartman, *Ordinary Differential Equations, 2nd. ed.*, SIAM Classics in Applied Mathematics 38, Society for Industrial and Applied Mathematics, Philadelphia, 2002; reprinted from 2nd. ed. Birkhäuser 1982; reprinted from original John Wiley & Sons, 1964.

This argument follows Chicone, who simplifies Hartman's proof. Details are very readable in Hartman's text. The argument in Perko and Liu, although based on the same idea, is a little harder and less transparent.

4. Outline.

- Setup.
- Change to good coordinates.
- Localizing the flow.
    - Flow exists to time one in small neighborhood.
    - Norms and Cutoff Functions.
    - Cutoff flow.
    - Inverting the time-one map.
- Hartman-Grobman Theorem.
    - Constructing candidate for topolological conjugacy of time-one map.
    - Key idea for proof.
    - Inverting generalized time-one maps.
    - Proof of homeomorphism.
    - Proof of local topological flow equivalence.

Take an open set $E \subset \mathbb{R}^n$, $x_0 \in E$, $f \in \mathcal{C}^1(E, \mathbb{R}^n)$ such that $f(x_0) = 0$. Let

$$A = Df[x_0].$$

Assume A is hyperbolic:

$$\Re e \lambda \neq 0 \qquad \text{for all eigenvalues of } A.$$

Consider the initial value problem

$$\begin{cases} \dot{y} = f(y); \\ y(0) = x. \end{cases}$$

Let the solution be denoted $\varphi(t, x)$.

Lemma (1.)

Let A be a real hyperbolic matrix. There is a linear change of variables that induces a splitting into stable and unstable spaces $\mathbb{R}^n = \mathcal{E}_s \oplus \mathcal{E}_u$ so that in the new variables

$$A = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}$$

and a constant $\alpha > 0$ so that for $t \geq 0$,

$$\begin{aligned} |e^{tA}x_s| &\leq e^{-\alpha t}|x_s|; \\ |e^{-tA}x_u| &\leq e^{-\alpha t}|x_u|; \end{aligned}$$

We have written $x_s = P_s x$, $x_u = P_u x$ where $P_s : \mathbb{R}^n \to \mathcal{E}_s$ and $P_u : \mathbb{R}^n \to \mathcal{E}_u$ are the orthogonal projections.

Let $\alpha > 0$ so that $|\Re e \lambda| > \alpha$ for all eigenvalues of A. There is a real invertible change of variables $Py = x$ that splits $\mathbb{R}^n$ into stable and unstable spaces and puts the matrix into Real Canonical Form

$$P^{-1}AP = \begin{pmatrix} J_1 & 0 & 0 & \cdots \\ 0 & J_2 & 0 & \cdots \\ 0 & 0 & J_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = D + N; \;\; J_i = \begin{pmatrix} R_i & I_i & 0 & \dots \\ 0 & R_i & I_i & \dots \\ 0 & 0 & R_i & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $R_i = \lambda_i$ and $I_i = 1$ for real eigenvalues and $R_k = \left(\begin{smallmatrix} a_k & b_k \\ -b_k & a_k \end{smallmatrix}\right)$ and $I_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for conjugate pairs of complex eigenvalues $\lambda_k = a_k \pm i b_k$ and D is block diagonal of all R_i's. Then change variables $Qz = y$ by $Q = \mathrm{diag}[S_1, S_2, \ldots]$ block diagonal with $S_i = \mathrm{diag}[1, \delta, \delta^2, \ldots]$ for each real eigenvalue block J_i of and $S_i = \mathrm{diag}[1, 1, \delta, \delta, \delta^2, \delta^2, \ldots]$ for each complex block J_i. Then in the z variable, the matrix has become

$$Q^{-1}P^{-1}APQ = D + \delta N = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}.$$

By choosing $\delta > 0$ small enough, we may arrange that

$$\Re e\lambda < -\alpha - \|\delta N\| \qquad \text{and} \qquad \Re e\kappa > \alpha + \|\delta N\|$$

for all eigenvalues λ of A_s and κ of A_u. Hence, for $t \geq 0$,

$$\begin{aligned} |e^{tA_s} z_s| &= |e^{t(D_s+\delta N_s)} z_s| \\ &= |e^{tD_s} e^{t\delta N} z_s| \\ &\leq \|e^{tD_s}\| \, \|e^{t\delta N}\| \, |z_s| \\ &\leq e^{-t(\alpha+\|\delta N\|)} e^{t\|\delta N_s\|} |z_s| \\ &= e^{-t\alpha}|z_s| \end{aligned}$$

and

$$\begin{aligned} |e^{-tA_u} z_u| &= |e^{-t(D_u+\delta N_u)} z_u| \\ &= |e^{-tD_u} e^{-t\delta N_u} z_u| \\ &\leq \|e^{-tD_u}\| \, \|e^{-t\delta N_u}\| \, |z_u| \\ &\leq e^{-t(\alpha+\|\delta N\|)} e^{t\|\delta N_u\|} |z_u| \\ &= e^{-t\alpha}|z_u|. \quad \square \end{aligned}$$

Let $D_\delta(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \le \delta\}$ denote the closed ball.

Lemma (2)

Let $E \subset \mathbb{R}^n$ be open, $x_0 \in E$ and $D_d(x_0) \subset E$ for some $d > 0$. Let $f \in \mathcal{C}^1(E, \mathbb{R}^n)$ such that $f(x_0) = 0$. Let $\varphi(t, x)$ denote the solution of

$$\begin{cases} \dot{y} = f(y); \\ y(0) = x. \end{cases}$$

Then there is $0 < c_1(f, d, x_0) < 1$ so that for all $0 < d_1 \le d$ and $x \in D_{c_1 d_1}(x_0)$, the solution $\varphi(t, x)$ exists for all $t \in [-1, 1]$ and satisfies $\varphi(t, x) \in D_{d_1}(x_0)$ whenever $(t, x) \in [-1, 1] \times D_{d_1}(x_0)$.

10. Proof of Lemma 2.

WLOG $x_0 = 0$. Let $M = \sup_{x \in D_d(x_0)} \|Df(x)\|$. Hence if $x \in D_d$, for the function $g(t) = f(tx)$,

$$|f(x)| = |g(1) - g(0)| = \left| \int_0^1 g'(s)\, ds \right| \leq$$

$$\leq \left| \int_0^1 Df(sx)[x]\, ds \right| \leq \int_0^1 \|Df(sx)\|\, |x|\, ds \leq M|x|.$$

Claim: $c_1 = e^{-M}$ works. If $x \in D_{c_1 d_1}$ then

$$|\varphi(t,x)| \leq |x| + \left| \int_0^t f(\varphi(s,x))\, ds \right| \leq |x| + M \int_0^t |\varphi(s,x)|\, ds.$$

By Gronwall's Inequality,

$$|\varphi(t,s)| \leq |x| e^{M|t|} \leq c_1 d_1 e^{M|t|} = d_1 e^{(|t|-1)M}$$

Thus for $|t| \leq 1$ the solution must exist since it stays within D_{d_1}. $\square$

For $u \in \mathcal{C}^1(E, \mathbb{R}^n)$, the norms

$$\|u\|_{\mathcal{C}^0(E)} = \sup_{x \in E} |u(x)|;$$

$$\|u\|_{\mathcal{C}^1(E)} = \|u\|_{\mathcal{C}^0(E)} + \|Du\|_{\mathcal{C}^0(E)}.$$

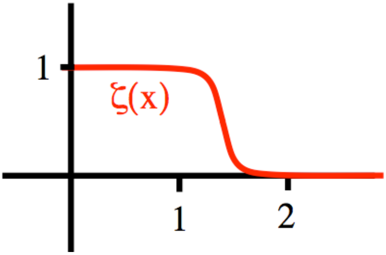

Figure: Cutoff Function.

Let $\zeta \in \mathcal{C}^\infty([0,\infty)$ such that $0 \le \zeta(t) \le 1$, $\zeta(t) = 1$ for $t \le 1$, $\zeta(t) = 0$ for $t \ge 2$ and $|\zeta'(t)| \le 2$. A cutoff function (or bump function) is $\beta_d(x) = \zeta\left(\frac{1}{d}|x|\right)$. Thus $\beta_d \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\beta_d(x) = \begin{cases} 1, & \text{if } 0 \le |x| \le d; \\ 0, & \text{if } |x| \ge 2d. \end{cases}$$

$$|D\beta_d(x)| \le \frac{2}{d}.$$

Lemma (3)

Let $E \subset \mathbb{R}^n$ be open, $x_0 \in E$ and $D_d(x_0) \subset E$ for some $d > 0$. Let $f \in \mathcal{C}^1(E, \mathbb{R}^n)$ such that $f(x_0) = 0$ and $A = Df(x_0)$. Let $\varphi(t, x)$ denote the flow by

$$\begin{cases} \dot{y} = f(y); \\ y(0) = x. \end{cases}$$

Let $d_1 \le d$ and c_1 from Lemma 2. For any $\alpha \in (0, 1)$, there is d_2 so that $0 < 2d_2 \le c_1 d_1$ and $p \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ so that $p(x_0) = 0$, $Dp(x_0) = 0$, $\|p\|_{\mathcal{C}^1(\mathbb{R}^n)} < \alpha$ and

$$\varphi_1(x) = \varphi(1, x) = e^A(x - x_0) + p(x)$$

fror all $x \in D_{d_2}(x_0)$.

13. Proof of Lemma 3.

WLOG $x_0 = 0$. Put $B = e^A$. Use the variation equation to estimate $L(t) = D_x\varphi(t, x)$.

$$\begin{cases} \dot{L} = Df(\varphi(t,x))[L], \\ L(0) = I. \end{cases}$$

But $\varphi(t, 0) = 0$ so that for all t, $\dot{L} = AL$ so $L(1) = e^{1 \cdot A} I = B$. Thus $D\varphi_1(0) = B$.

Cut off the nonlinear part of the flow to get

$$p(x) = \beta_{d_2}(x)\left(\varphi_1(x) - Bx\right).$$

$p(0) = \varphi_1(0) = 0$. Since $\beta_{d_2} \equiv 1$ in a neighborhood of 0, it follows that

$$Dp(0) = D\varphi_1(0) - B = 0.$$

Since φ is $\mathcal{C}^1$ in x, choose d_2 so small that $0 < 2d_2 < c_1 d_1$ and

$$\sup_{x \in D_{2d_2}} |D\varphi_1(x) - B| < \eta = \frac{\alpha}{2d_1 + 6}.$$

Thus

$$|p| = |\beta_{d_2}|\,|\varphi_1(x) - Bx| \leq 1 \cdot \eta |x| \leq 2\eta d_2$$

since $p(x) = 0$ for $|x| \geq 2d_2$.
We also have a derivative estimate:

$$\begin{aligned} |Dp| &\leq |D\beta_{d_2}|\,|\varphi_1 - Bx| + |\beta_{d_2}|\,|D\varphi_1 - B| \\ &\leq \frac{2}{d_2}\eta |x| + 1 \cdot \eta \\ &\leq \frac{2}{d_2}\eta \cdot 2d_2 + 1 \cdot \eta = 5\eta. \end{aligned}$$

Thus

$$\|p\|_{\mathcal{C}^1(\mathbb{R}^n)} \leq \|p\|_{\mathcal{C}^0(\mathbb{R}^n)} + \|Dp\|_{\mathcal{C}^0(\mathbb{R}^n)} \leq 2\eta d_2 + 5\eta < \alpha. \quad \square$$

Lemma (4)

Let $T(x) = Bx + p(x)$ where B is invertible and $p \in \mathcal{C}^1(\mathbb{R}^n.\mathbb{R}^n)$ such that $p(0) = 0$ and

$$\|p\|_{\mathcal{C}^1(\mathbb{R}^n)} \le \gamma = \frac{1}{2\|B^{-1}\|}.$$

Then there is $T^{-1} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ such that $\|T^{-1}\| \le 2\|B^{-1}\|$.

Proof. It suffices to find a continuous T^{-1}. $\mathcal{C}^1$ follows from the Inverse Function Theorem. For all $y \in \mathbb{R}^n$, solve for x in $y = Bx + p(x)$. Formulate as a fixed point

$$x = g(x) = B^{-1}(y - p(x)).$$

$g(x)$ is a contraction in $\mathbb{R}^n$. First, for $x, z \in \mathbb{R}^n$,

$$|p(x) - p(z)| \le \gamma |x - z|$$

so

$$|p(x)| = |p(x) - p(0)| \le \gamma |x - 0| = \gamma |x|.$$

Thus

$$
\begin{aligned}
|g(x)-g(z)| &= |B^{-1}(p(x)-p(z))| \\
&\le \|B^{-1}\||p(x)-p(z)| \\
&\le \|B^{-1}\|\gamma|x-z| \le \tfrac{1}{2}|x-z|.
\end{aligned}
$$

Thus iterating with $x_0 = 0$, $x_1 = g(0) = B^{-1}y$ and $x_{n+1} = g(x_n)$, there is a unique fixed point

$$x = T^{-1}(y) = x_0 + \sum_{i=0}^{\infty}(x_{i+1} - x_i).$$

Hence

$$|T^{-1}(y)| \le \sum_{i=0}^{\infty} \frac{1}{2^i}|B^{-1}y| \le 2\|B^{-1}\||y|.$$

Also writing $x_i = T^{-1}(y_i)$,

$$y_1 - y_2 = B(x_1 - x_2) + p(x_1) - p(x_2)$$

so

$$\begin{aligned} |x_1 - x_2| &\le |B^{-1}(y_1 - y_2)| + |B^{-1}(p(x_1) - p(x_2))| \\ &\le \|B^{-1}\||y_1 - y_2| + \tfrac{1}{2}|x_1 - x_2|. \end{aligned}$$

Thus, we have an esrimate in terms of itself, a “GARBAGE TRUCK”

$$|T^{-1}(y_1) - T^{-1}(y_2)| \le 2\|B^{-1}\||y_1 - y_2|$$

the inverse is Lipschitz continuous and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.

To check that T^{-1} is $\mathcal{C}^1$ we check that the differential of T is invertible at any point. Thus, for any $y \in \mathbb{R}^n$ and corresponding $x = T^{-1}(y)$,

$$DT(x) = B + Dp(x) = B(I + B^{-1}Dp(x)).$$

It is invertible because $\|B^{-1}Dp(x)\| \leq \|B^{-1}\|\gamma \leq \frac{1}{2}$: in fact, the inverse of $I + M$ where $\|M\| \leq \frac{1}{2}$ is given by the convergent series

$$(I + M)^{-1} = \sum_{k=0}^{\infty}(-1)^k M^k.$$

Thus the inverse function $T^{-1}(y)$ is $\mathcal{C}^1$ in a neighborhood of y by the Inverse Function Theorem and $DT^{-1}(y) = (DT(x))^{-1}$. $\square$

Theorem (Hartman Grobman)

Let $E \subset \mathbb{R}^n$ be open, $x_0 \in E$ and $D_d(x_0) \subset E$ for some $d > 0$. Let $f \in \mathcal{C}^1(E, \mathbb{R}^n)$ such that $f(x_0) = 0$ and $A = Df(x_0)$ is hyperbolic. Then there are open neighborhoods $x_0 \in U$ and $0 \in V$ and a homeomorphism $H : U \to V$ such that the flow $\varphi(t, x)$ of the equation

$$\begin{cases} \dot{y} = f(y), \\ y(0) = x \end{cases}$$

is topologically flow equivalent to the flow of its linearization $e^{tA}x$:

$$H(\varphi(t, x)) = e^{tA} H(x)$$

for all $x \in U$ and $|t| \leq 1$.

Combining with the theorem for linear flows, this implies that two hyperbolic systems are locally topologically flow equivalent if and only if their stable spaces have equal dimensions.

The proof is to show that the time-one maps are locally topologically conjugate and then recover the flow equivalence. The proof will be completed by Lemmas 5-8.

WLOG $x_0 = 0$. We assume $\dim \mathcal{E}_s \geq 1$ and $\dim \mathcal{E}_e \geq 1$ throughout. The remaining cases are similar and simpler. Using Lemma 1, we change to good coordinates. Let

$$B = e^A = \begin{pmatrix} e^{A_s} & 0 \\ 0 & e^{A_u} \end{pmatrix}.$$

Define

$$T(x) = Bx + p(x)$$

as in Lemma 3, so $T = \varphi_1$ in D_{d_2}. We seek a homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$H \circ T = B \circ H.$$

We begin with a lemma that produces a candidate for the topological conjugacy.

Lemma (5)

Let A be hyperbolic and $B = e^A$. Let $p \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfy $\|p\|_{\mathcal{C}^1} < \gamma = \frac{1}{2}\|B^{-1}\|$. Let $T(x) = Bx + p(x)$. Then there is a unique continuous $H : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$H(T(x)) = BH(x). \tag{1}$$

Define the Banach space of bounded continuous maps:

$$X = \{h \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n) : \|h\|_{\mathcal{C}^0} < \infty\}.$$

We shall look for $H(x) = x + h(x)$ where $h \in X$.

22. Proof of Lemma 5.

Assume that we are in good coordinates and have $\alpha > 0$ as given by Lemma 1. Using $H(x) = x + h(x)$, and $T(x) = Bx + p(x)$, (1)

$$T(x) + h(T(x)) = BT(x)$$

becomes

$$Bx + p(x) + h(T(x)) = Bx + Bp(x)$$

so

$$h(T(x)) = Bh(x) - p(x).$$

Split the equation into stable and unstable parts

$$\begin{cases} h_s(T(x)) = e^A h_s(x) - p_s(x), & e^A \text{ is a contraction on } \mathcal{E}_s; \\ h_u(T(x)) = e^A h_u(x) - p_u(x), & e^A \text{ is an expansion on } \mathcal{E}_u; \end{cases}$$

KEY TRICK: invert the second equation to make it a contraction too!

By Lemma 4, we may use the continuously differentiable inverse T^{-1}

$$\begin{cases} h_s(x) = e^A h_s(T^{-1}(x)) - p_s(T^{-1}(x)) & = G_s(h)(x), \\ h_u(x) = e^{-A} h_u(T(x)) + e^{-A} p_u(x) & = G_u(h)(x). \end{cases} \tag{2}$$

We will solve this functional equation using the Contraction Mapping principle in X.

Using the norm on $x \in \mathbb{R}^n$

$$|x|_m = \max(|x_s|, |x_u|)$$

and corresponding sup-norm on $h \in X$

$$\|h\|_X = \sup_{x \in \mathbb{R}^n} |h(x)|_m$$

we show that (2) is a contraction.

Assume that we are in good coordinates and have $\alpha > 0$ as given by Lemma 1. For maps $g, h \in X$, we have

$$
\begin{aligned}
|G(g) - G(h)|_m &\leq \max\Big(|e^{A_s} g_s(T^{-1}(x)) - e^{A_s} h_s(T^{-1}(x))|, \\
&\qquad\qquad |e^{-A_u} g_u(T(x)) - e^{-A_u} h_u(T(x))|\Big) \\
&\leq e^{-\alpha} \max\left(\|g_s - h_s\|_{\mathcal{C}^0(\mathbb{R}^n)}, \|g_u - h_u\|_{\mathcal{C}^0(\mathbb{R}^n)}\right) \\
&\leq e^{-\alpha} \|g - h\|_m
\end{aligned}
$$

Thus, by the Contraction Mapping principle, there is a map $h \in X$ satisfying (2). $\square$

We recover Lemma 5 by taking $p = 0$ in the following. It will be used to construct an inverse to H from Lemma 5.

Lemma (6)

Let A be hyperbolic, $B = e^A$ and let $\alpha > 0$ be the constant from Lemma 1. Let $p, q \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfy

$$\|p\|_{\mathcal{C}^1} = \gamma < \min\left(\tfrac{1}{2}\|B^{-1}\|, e^\alpha - 1\right)$$

and

$$\|q\|_{\mathcal{C}^1} < \tfrac{1}{2}\|B^{-1}\|.$$

Let $T(x) = Bx + p(x)$ and $S(x) = Bx + q(x)$. Then there is a unique continuous $G : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$T(G(x)) = G(S(x)). \tag{3}$$

The proof is very similar to that of Lemma 5 so will not be given here.

Lemma (7)

Let A be hyperbolic and $B = e^A$. Let $p \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfy

$$\|p\|_{\mathcal{C}^1} = \gamma < \min\left(\tfrac{1}{2}\|B^{-1}\|, e^{\alpha} - 1\right).$$

Let $T(x) = Bx + p(x)$. Then the map $H : \mathbb{R}^n \to \mathbb{R}^n$ from Lemma 5 satisfying

$$H \circ T(x) = BH(x)$$

for all $x \in \mathbb{R}^n$ is a homeomorphism.

Proof. The idea is to construct a continuous inverse $K \in \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n)$ such that $H \circ K = I$ and $K \circ H = I$. Taking $p = 0$ in Lemma 6, there is $k \in X$ such that $K(x) = x + k(x)$ satisfies

$$T(K(y)) = K(By).$$

But $H \circ T = BH$, thus for all $y \in \mathbb{R}^n$

$$H \circ K(By) = H \circ T \circ K(y) = BH \circ K(y). \tag{4}$$

Thus $G = H \circ K$ satisfies (3) in Lemma 6 with $p = q = 0$. Thus there is $v \in X$ such that for all $y \in \mathbb{R}^n$,

$$y + k(y) + h(y + k(y)) = H \circ K(y) = y + v(y).$$

But $G = I$ also satisfies (4). By uniqueness,

$$H \circ K = I.$$

Similarly, by Lemmas 5 and 6,

$$K \circ H \circ T(x) = K \circ BH(x) = T \circ K \circ H(x). \tag{5}$$

This time, $K \circ H$ satisfies (3) of Lemma 6 with $q = p$. Thus there is $w \in X$ such that for all $x \in \mathbb{R}^n$,

$$x + h(x) + k(x + h(y)) = K \circ H(x) = x + w(y).$$

But I also satisfies (5). Thus by uniqueness,

$$K \circ H = I. \quad \square$$

Lemma (8)

Let $d_2 > 0$ be chosen so small in Lemma 3 such that p satisfies conditions of Lemma 7. Let H be the map constructed in Lemma 5 which by Lemma 7 is a topological conjugacy

$$H \circ T = e^A H.$$

For $x \in D_{d_2}$ this is a topological conjugacy between time-one flows:

$$H \circ \varphi_1(x) = e^A H(x).$$

Then the map

$$\mathcal{H}(x) = \int_0^1 e^{-\sigma A} H(\varphi(\sigma, x))\, d\sigma$$

is a local topological flow equvalence: for all $(t, x) \in [-1, 1] \times D_{d_2}$:

$$\mathcal{H}(\varphi(t, x)) = e^{tA} \mathcal{H}(x).$$

Moreover $H = \mathcal{H}$ on D_{d_2}.

30. Proof of Lemma 8.

Let's do the case $x \in D_{d_2}$ and $t \geq 0$. The case $t \leq 0$ is similar. Use the semigroup property $\varphi_\sigma \circ \varphi_t = \varphi_{\sigma+t}$,

$$\begin{aligned} e^{-tA}\mathcal{H}\varphi_t &= \int_0^1 e^{-tA} e^{-\sigma A} H \circ \varphi_\sigma \circ \varphi_t \, d\sigma \\ &= \int_0^1 e^{-(t+\sigma)A} H \circ \varphi_{\sigma+t} \, d\sigma. \end{aligned}$$

Change variables $\tau = \sigma + t - 1$, use $B = e^A$ and $H \circ \varphi_1 = BH$.

$$\begin{aligned} &= \int_{t-1}^{t} e^{-(1+\tau)A} H \circ \varphi_{1+\tau} \, d\tau \\ &= \int_{t-1}^{t} e^{-\tau A} B^{-1} H \circ \varphi_1 \circ \varphi_\tau \, d\tau \\ &= \int_{t-1}^{t} e^{-\tau A} H \varphi_\tau \, d\tau \end{aligned}$$

Split integral and change variables in the first integral $\tau = \sigma - 1$,

$$\begin{aligned}
e^{-tA}\mathcal{H}\varphi_t &= \int_{t-1}^{0} e^{-\tau A} H\varphi_\tau \, d\tau + \int_0^t e^{-\tau A} H\varphi_\tau \, d\tau \\
&= \int_t^1 e^{-\sigma A} e^{A} H\varphi_{\sigma-1} \, d\sigma + \int_0^t e^{-\tau A} H\varphi_\tau \, d\tau \\
&= \int_t^1 e^{-\sigma A} H\varphi_1 \circ \varphi_{\sigma-1} \, d\sigma + \int_0^t e^{-\tau A} H\varphi_\tau \, d\tau \\
&= \int_t^1 e^{-\sigma A} H\varphi_\sigma \, d\sigma + \int_0^t e^{-\tau A} H\varphi_\tau \, d\tau \\
&= \int_0^1 e^{-\tau A} H\varphi_\tau \, d\tau \\
&= \mathcal{H}.
\end{aligned}$$

Thus $\mathcal{H} \circ \varphi_1 = e^{tA}\mathcal{H}$. When $t = 1$, then $\mathcal{H}$ satisfies $\mathcal{H} \circ T = B\mathcal{H}$. By Lemma 5 uniqueness, $\mathcal{H} = H$. Thus, $\mathcal{H}$ is a local homeomorphism. □

Thanks!