Homework for Math 6410 §1, Fall 2017

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Our main text this semester is Thomas Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press 2013, ISBN 978-94-6239-020-1. Please read the relevant sections in the text as well as any cited reference. Each problem is due three class days after its assignment, or on Tuesday, Dec. 12, whichever comes first.

1. [Aug. 24] **Compute a Phase Portrait using the Computer**

   This exercise asks you to figure out how to make a computer algebra system draw a phase portrait. For many of you this will already be familiar. See, *e.g.,* the MAPLE worksheet from today's lecture

$$\text{http} : //\text{www.math.utah.edu}/\sim\text{treiberg/M6412eg1.mws}$$
$$\text{http} : //\text{www.math.utah.edu}/\sim\text{treiberg/M6412eg1.pdf}$$

   or my lab notes from Math 2280,

$$\text{http} : //\text{www.math.utah.edu}/\sim\text{treiberg/M2282L4.mws.}$$

   Choose an autonomous system in the plane with at least two rest points such that one of the rest points is a saddle and another is a source or sink. (Everyone in class should have a different ODE.) Compute the Jacobian at the rest points. Find the eigenvalues of the Jacobians and determine the stability type at each rest point. Using your favorite computer computer algebra system, *e.g.,* MAPLE or MATLAB or any ODE solver, plot the phase portrait indicating the background vector field and enough integral curves to show the topological character of the flow. You should include the stable and unstable curves at the saddles, trajectories at the all rest points including any that connect the nodes, as well as any separatrices.

2. [Aug. 23] **Matrix Exponential**

   (a) Write the second order equation

$$\ddot{u} + 2\dot{u} + u = 0 \tag{1}$$

   as a first order system, $\dot{x} = Ax$.

   (b) Find e^{tA} for the matrix of part (a).

   (c) Use the answer from part (b) to find the solution of (1) with $u(0) = u_0$ and $\dot{u}(0) = u_1$.

   [*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press, 2013, p.18.]

3. [Aug 25] **Jordan Form.**

   Find the eigenvalues, generalized eigenvectors, cyclic vector chains and S such that $S^{-1}AS = J$, the Jordan form.

$$
A = \begin{pmatrix}
5 & 10 & 9 & 3 & 0 & 5 \\
-1 & -4 & -1 & -1 & 0 & 0 \\
6 & 7 & 4 & 1 & 0 & 5 \\
-11 & -12 & -11 & -3 & 0 & -10 \\
34 & 44 & 39 & 11 & -2 & 26 \\
-7 & -10 & -9 & -3 & 0 & -7
\end{pmatrix}
$$

   [In MAPLE,

   `Matrix(6, 6, [[5, 10, 9, 3, 0, 5], [-1, -4, -1, -1, 0, 0], [6, 7, 4, 1, 0, 5],`
   `[-11, -12, -11, -3, 0, -10], [34, 44, 39, 11, -2, 26], [-7, -10, -9, -3, 0, -7]])`

   ]

4. [Aug. 28] **Real Canonical Form.**

   Let A be a real 2×2 matrix whose eigenvalues are $a \pm bi$ where $a, b \in \mathbb{R}$ such that $b \neq 0$. Show that there is a real matrix Q so that

$$
Q^{-1}AQ = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.
$$

   Use this fact to solve the system

$$
\begin{aligned}
x' &= -13x - 10y \\
y' &= 20x + 15y
\end{aligned}
$$

5. [Aug. 30] **Just Multiply by t.**

   Consider the n-th order constant coefficient linear homogeneous scalar equation

$$
x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0
$$

   where a_i are complex constants. Convert to a first order differential system $\mathbf{x}' = A\mathbf{x}$. Show that the geometric multiplicity of every eigenvalue of A is one. Show that a basis of solutions is $\{t^k \exp(\mu_i t)\}$ where $i = 1, \ldots, s$ correspond to the distinct eigenvalues μ_i and $0 \leq k < m_i$ where m_i is the algebraic multiplicity of μ_i. [*cf.*, Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems*, Amer. Math. Soc., 2012, p.68.]

6. [Aug. 31] **To Use Jordan Form or Not to Use Jordan Form.**

   Sometimes the use of the Jordan Canonical Form and matrices with multiple eigenvalues can be avoided using the following considerations. Let $A \in \mathcal{L}(\mathbb{C}^n)$.

   (a) Show for any $\epsilon > 0$ there is a matrix B with distinct eigenvalues so that $\|A - B\| \leq \epsilon$.

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(b) By a simpler algorithm than finding the Jordan Form, one can change basis by a P that transforms A to upper triangular, called the Schur Form of the matrix.

$$P^{-1}AP = U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}. \tag{2}$$

Show that this fact can be used instead of Jordan Form to characterize all solutions of $\dot{y} = Ay$ (as linear combinations of products of certain exponentials, polynomials and trigonometric functions). [*c.f.*, Bellman, *Stability Theory of Differential Equations*, pp. 21–25.])

(c) Show that given $\epsilon > 0$ there exists a nonsingular P such that in addition to (2) we may arrange that $\sum_{i<j} |u_{ij}| < \epsilon$.

(d) Give three proofs of $\det(e^A) = e^{\text{trace}(A)}$.

(e) [Optional] Find all continuous scalar valued functions $f : \mathcal{L}(\mathbb{C}^n) \to \mathbb{C}$ so that

$$f(AB) = f(A)f(B) \qquad \text{for all } A, B.$$

You can probably find several different arguments on your own. [*ibid.*; or Kurosh, *Higher Algebra*, p. 334.]

7. [Sept. 6] **Liapunov Stability for Linear Equations.**

Let A be an $n \times n$ real matrix. Find necessary and sufficient conditions on A so that for all $x_0 \in \mathbf{R}^n$, the solution $\varphi(t; x_0)$ of (3) remains bounded for $t \geq 0$. Prove your result.

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0. \end{cases} \tag{3}$$

Show that boundedness of all solutions for $t \geq 0$ is equivalent to $\bar{x} = 0$ being Liapunov Stable. [University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2017.]

8. [Sept. 8] Let A and B be 2×2 real matrices.

(a) Prove the following statements about solutions of $\dot{z} = Az$.

   i. If $\operatorname{tr} A < 0$ and $\det A > 0$ then the origin is asymptotically stable.
   ii. If $\operatorname{tr} A < 0$ and $\det A = 0$ then the origin is Liapunov stable.
   iii. If $\operatorname{tr} A = 0$ and $\det A > 0$ then the origin is Liapunov stable.
   iv. In all other cases the origin is unstable.

(b) If the origin is asymptotically stable for $\dot{z} = Az$, prove that there exists $\delta > 0$ (depending on A) such that if $\|A - B\| < \delta$ then the origin is asymptotically stable for $\dot{z} = Bz$.

(c) Is the preceding statement true if "asymptoticlly stable" is replaced by "Liapunov stable"?

[*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems,* Atlantis Press, 2013, pp. 50–51.]

9. [Sept. 11] **Delay Differential Equation.**

The delay differential equation involves past values of the unknown function x, and so its initial data φ must be given for all times $t \leq 0$. Using the Contraction Mapping Principle, show the local existence of a solution to the delay differential equation.

Theorem. Let $f \in \mathcal{C}(\mathbf{R}^3)$ be a function that satisfies a Lipschitz condition: there is $L < \infty$ such that for all $t, x_1, x_2, y_1, y_2 \in \mathbf{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|).$$

Let $g \in \mathcal{C}(\mathbf{R})$ such that $g(t) \leq t$ for all t. Let $\varphi \in \mathcal{C}\big((-\infty, 0], \mathbf{R}\big)$. Show that there is an $r > 0$ such that the initial value problem

$$\begin{cases} \dfrac{dx}{dt}(t) = f\big(t, x(t), x(g(t))\big) & \text{for } 0 < t < r, \\ x(t) = \varphi(t) & \text{for all } t \leq 0 \end{cases}$$

has a unique solution $x(t) \in \mathcal{C}\big((-\infty, r], \mathbf{R}\big) \cap \mathcal{C}^1\big((0, r), \mathbf{R}\big)$.

[*cf.* Saaty, Modern Nonlinear Equations, Dover 1981, §5.5.]

10. [Sept. 13] **Alternative Proof of the Picard Theorem.**

We set up the same as in the Picard Theorem. Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be an open set and $(t_0, x_0) \in \Omega$. Suppose $f(t, x) \in \mathcal{C}(\Omega, \mathbf{R}^n)$ satisfies a local Lipschitz condition in x uniformly in t. Let $a, b > 0$ so that the closed cylinder $\Sigma = [t_0 - a, t_0 + a] \times \overline{B_b(x_0)} \subset \Omega$. Let

$$M = \sup_{(t,x) \in \Sigma} |f(t, x)|$$

and L be the Lipschitz constant on Σ. Let $\epsilon = \min\{a, b/M\}$ and $I = [t_0 - \epsilon, t_0 + \epsilon]$. Let $X = \mathcal{C}(I, \mathbf{R}^n)$ be the Banach Space of continuous curves with sup-norm $\| \bullet \|$. Then $C = \{y \in X : |y(t) - x_0| \leq b \text{ for all } t \in I\}$ is a closed subset. For $y \in C$ we defined the operator

$$T[y](t) = x_0 + \int_{t_0}^{t} f\big(s, y(s)\big)\, ds.$$

We have shown that $T : C \to C$. Put $y_0 = x_0$ and let $y_{n+1} = T(y_n)$ for all n. We have $y_n \in C$ for all n by induction.

(a) Prove

$$|y_{n+1}(t) - y_n(t)| \leq \frac{ML^n |t - t_0|^{n+1}}{(n+1)!} \qquad \text{for all } n = 0, 1, 2, \ldots \text{ and } t \in I.$$

(b) Using (a), show that there is a solution $x \in \mathcal{C}^1(I, \mathbf{R}^n)$ of the initial value problem

$$\dot{x}(t) = f\big(t, x(t)\big), \qquad \text{for } t \in I;$$
$$x(t_0) = x_0.$$

Note that this improves our result using contraction mapping, exploiting the fact that the integral equation is of Volterra type.

11. [Sept. 15] **Linear Equations are Benign.**

Let the $n \times n$ matrix $A(t)$ and the vector $b(t) \in \mathbf{R}^n$ be continuous functions for all $t \in \mathbf{R}$. Show that solutions of the inhomogeneous linear equation

$$\begin{cases} \dfrac{dx}{dt}(t) = A(t)x(t) + b(t) \\ x(t_0) = x_0 \end{cases}$$

are bounded on finite intervals so exist for all time.

12. [Sept. 18] **Differentiability of Fixed Points.**

Let $f(x, \epsilon) \in \mathcal{C}^1(\mathbf{R}^2, \mathbf{R})$ be a continuously differentiable contraction: there is a $\lambda \in (0, 1)$ so that
$$|f(x, \epsilon) - f(y, \epsilon)| \leq \lambda |x - y| \qquad \text{for all } x, y, \epsilon.$$

Show that the unique fixed point $g(\epsilon)$ which satisfies $g(\epsilon) = f(g(\epsilon), \epsilon)$ is a continuously differentiable function of ϵ.

13. [Sept. 20] **Existence via Schauder's Theorem.**

Give another proof of the Peano Existence Theorem using the Schauder Fixed Point Theorem.

Theorem. [Peano Existence Theorem] Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be a domain and $f \in \mathcal{C}(\Omega, \mathbf{R}^n)$. Then for any $(t_0, x_0) \in \Omega$ there is $\epsilon > 0$ and a continuously differentiable function $x(t) : [t_0 - \epsilon, t_0 + \epsilon] \to \mathbf{R}^n$ which solves the initial value problem

$$\frac{dx}{dt} = f(t, (x(t))), \qquad \text{for all } t \in [t_0 - \epsilon, t_0 + \epsilon];$$
$$x(t_0) = x_0.$$

Theorem. [Schauder Fixed Point Theorem] Let $\mathcal{A}$ be a closed, bounded, convex subset of a Banach space $\mathcal{X}$ and $T : \mathcal{A} \to \mathcal{A}$ be a completely continuous function. Then T has a fixed point in $\mathcal{A}$.

A subset $\mathcal{K}$ of a Banach space is *compact* if any sequence in $\{\phi_i\}_{i=1,2,\dots} \subset \mathcal{K}$ has a subsequence that converges to an element in $\mathcal{K}$. f is *compact* if for any bounded set $\mathcal{B} \subset \mathcal{X}$, the closure of the set $f(\mathcal{B})$ is compact. f is *completely continuous* if it is both compact and continuous. [*cf.* Hale, p. 14.]

14. [Sept. 22.] **Compare Solutions of Two Mathieu Equations.**

One solution for Problem 19 of the 2010 Math 6410 depended on comparing the solutions of the perturbed and unperturbed problems. Find a sharp estimate for the difference in values and derivatives at $T = \frac{2\pi}{3}$ of the solutions for the two initial value problems, where u_0, u_1, ϵ are constants.

$$\ddot{x} + x = 0, \qquad\qquad\qquad \ddot{y} + (1 + \epsilon \sin(3t))y = 0,$$
$$x(0) = u_0, \qquad\qquad\qquad\qquad y(0) = u_0,$$
$$\dot{x}(0) = u_1; \qquad\qquad\qquad\qquad \dot{y}(0) = u_1.$$

Let $y(t; u_0, u_1, \epsilon)$ solve the IVP. Use your estimate to show $|y(T; 1, 0, \epsilon) + \dot{y}(T; 0, 1, \epsilon)| < 2$ for small $|\epsilon|$.

15. [Sept. 25.] **Escape Times.**

Show that each solution $(x(t), y(t))$ of the initial value problem

$$\begin{cases} x' = x^2 + y \\ y' = y^2 + x \end{cases} \qquad \begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

with $x_0 > 0$ and $y_0 > 0$ cannot exist on an interval of the form $[0, \infty)$.

[*cf.* Wilson, *Ordinary Differential Equations,* Addison-Wesley, 1971, p.255.]

16. [Sept. 27.] **Variation of Parameters Formula.**

Solve the inhomogeneous linear system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\,\mathbf{x} + \mathbf{b}(t), \\ \quad\ \mathbf{x}(t_0) = \mathbf{c}; \end{cases}$$

where

$$A(t) = \begin{pmatrix} -2\cos^2 t & -1 - \sin 2t \\ 1 - \sin 2t & -2\sin^2 t \end{pmatrix}, \qquad \mathbf{b}(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Hint: a fundamental matrix is given by

$$U(t,0) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}.$$

[*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 62.]

17. [Sept. 29.] **Application of Liouville's Theorem.**

Find a solution of the IVP for Bessel's Equation of order zero

$$\begin{cases} x'' + \dfrac{1}{t}\,x' + x = 0 \\ x(0) = 1, \qquad x'(0) = 0 \end{cases}$$

by assuming the solution has a power series representation (or use Frobenius Method.) Use Liouville's formula for the Wronskian to find a differential equation for a second linearly independent solution of the differential equation. Show that this solution blows up like $\log t$ as $t \to 0$. [*cf.* Fritz John, Ordinary Differential Equations, Courant Institute of Mathematical Sciences, 1965, p. 90.]

18. [Oct. 2.] **Condition for Asymptotic Stability.**

Suppose that the zero solution to $\dot{x} = Ax$ is asymptotically stable. Let $g(t,x) \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ satisfy $g(t,0) = 0$ and

$$|g(t,x)| \le h(t)|x|, \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n,$$

where $h(t)$ satisfies for positive constants k and r,

$$\int_0^t h(s)\,ds \le kt + r, \qquad \text{for all } t \ge 0.$$

Show that there is a constant $k_0(A) > 0$ such that if $k \le k_0$, then the zero solution of

$$\dot{x} = Ax + g(t,x)$$

is asymptotically stable. [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations,* Prentice Hall 2003, p. 243.]

19. [Oct. 4.] **Linearized Stability of Fixed Points.**

    The SIR model of epidemics of Brauer and Castillo-Chávez relates three populations, $S(t)$ the susceptible population, $I(t)$ the infected population and $R(t)$ the recovered population. The other variables are positive constants. Assume that births in the susceptible group occur at a constant rate μK. Assume that there is a death rate of $-\mu$ for each population. Assume also that there is an infection rate of people in the susceptible population who become infected which is proportional to the contacts between the two groups βSI. There is a recovery of γI from the infected group into the recovered group. Finally, the disease is fatal to some in the infected group, which results in the removal rate $-\alpha I$ from the infected population. The resulting system of ODE's is

    $$\dot{S} = \mu K - \beta SI - \mu S$$
    $$\dot{I} = \beta SI - \gamma I - \mu I - \alpha I$$
    $$\dot{R} = \gamma I - \mu R$$

    (a) Note that the first two equations decouple and can be treated as a 2×2 system. Then the third equation can be solved knowing $I(t)$. Let $\delta = \alpha + \gamma + \mu$. For the 2×2 system, find the nullclines and the fixed points.

    (b) Check the stability of the nonnegative fixed points. Show that for $\beta K < \delta$ the disease dies out. Sketch the nullclines and some trajectories in the phase plane in this case.

    (c) Show that for $\beta K > \delta$ the epidemic reaches a steady state. Sketch the nullclines and some trajectories in the phase plane now.

    [From R. C. Robinson, *An Introduction to Dynamical Systems*, Pearson 2004.]

20. [Oct. 6.] **Liapunov Functions.**

    Use a Liapunov Function to show that the zero solution is asymptotically stable

    $$\ddot{x} + (2 + 3x^2)\,\dot{x} + x = 0.$$

    Hint: A sneaky way is to show that this equation is equivalent to the system

    $$\dot{x} = y - x^3$$
    $$\dot{y} = -x + 2x^3 - 2y.$$

    [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 348–349.]

21. [Oct. 16.] **La Salle's Invariance Principle.** Assume that the continuously differentiable functions satisfy $f(x) > 0$ and $xg(x) > 0$ for $x \neq 0$. Use La Salle's Invariance Principle to show that the zero solution of
    $$\ddot{x} + f(x)\,\dot{x} + g(x) = 0$$
    is asymptotically stable. [Brauer & Nohel, *The Qualitative Theory of Ordinary Differential Equations: An Introduction.*, Dover, 1989.]

22. [Oct. 18.] **Četaev's Theorem.**

    Show that the zero solution is not stable.

    $$\dot{x} = x^3 + xy$$
    $$\dot{y} = -y + y^2 + xy - x^3.$$

    [J. Hale and H. Koçek, *Dynamics and Bifurcations*, Springer 1991, p. 286.]

23. [Oct. 20.] **Asymptotically Stable Equilibrium in a Discrete Dynamical System.**

Let $T \in \mathcal{C}(\mathbf{R}^n, \mathbf{R}^n)$. Consider the nonlinear difference equation

$$x_0 = x,$$
$$x_{n+1} = T(x_n) \qquad \text{for } n \geq 0. \tag{4}$$

Writing $Tx := T(x)$, a solution sequence of (4) can be given as the n-th iterates $x_n = T^n x$ where $T^0 = I$ is the identity function and $T^n = TT^{n-1}$. The solution automatically exists and is unique on nonnegative integers $\mathbf{Z}_+$. Solutions $T^n x$ depend continuously on x since T is continuous. The *forward orbit* of a point x is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. A set $H \subset \mathbf{R}^n$ is *positively (negatively) invariant* if $T(H) \subset H$ ($H \subset T(H)$). H is said to be *invariant* if $T(H) = H$, that is if it is both positively and negatively invariant. The solution $T^n x$ starting from a given point x is *periodic* or *cyclic* if for some $k > 0$, $T^k x = x$. The least such k is called the *period* of the solution or the *order* of the cycle. If $k = 1$ then x is a *fixed point* of T or an *equilibrium state* of (4).

(a) Let A be a real $n \times n$ matrix such that $|\lambda| < \gamma$ for all eigenvalues λ of A. Show that there is a norm $\| \cdot \|$ on $\mathbf{R}^n$ so that $\|Ax\| \leq \gamma \|x\|$ for all $x \in \mathbf{R}^n$.

(b) Let $P \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ such that $P(0) = 0$ and $|\lambda| < 1$ for all eigenvalues of $DP(0)$. Show that 0 is an asymptotically stable fixed point of the discrete dynamical system in $\mathbf{R}^n$

$$x_1 = x,$$
$$x_{n+1} = P(x_n).$$

24. [Oct. 23.] T-**Periodic Linear Equations.**

Consider the T-periodic non-autonomous linear differential equation

$$\dot{x} = A(t)\,x, \qquad x \in \mathbf{R}^n, \qquad A(t+T) = A(t).$$

Let $X(t)$ be the fundamental matrix with $X(0) = I$.

(a) Show that there is at least one nontrivial solution $\chi(t)$ such that $\chi(t+T) = \rho\chi(t)$, where ρ is an eigenvalue of $X(T)$.

(b) Suppose that $X(T)$ has n distinct eigenvalues ρ_i, $i = 1, \ldots, n$. Show that there are n linearly independent solutions of the form $x_i = p_i(t)e^{\nu_i t}$ where $p_i(t)$ is T-periodic. How is ρ_i related to ν_i?

(c) Consider the equation $\dot{x} = f(t)\,A_0\,x$, $x \in \mathbf{R}^2$, with $f(t)$ a scalar T-periodic function and A_0 a constant matrix with real distinct eigenvalues. Determine the corresponding Floquet Multipliers.

[U. Utah PhD Preliminary Examination in Differential Equations, August 2008.]

25. [Oct. 25.] **Blowup in a Periodic Linear Equation.**

Let $\phi(t)$ be a real, continuous, π-periodic function. Consider the scalar equation

$$\ddot{x} - (\cos^2 t)\dot{x} + \phi(t)\,x = 0.$$

Show that there is a real solution that tends to infinity as $t \to \infty$.

[*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations*, Prentice Hall 2003, p. 162.]

26. [Oct. 27.] **Boundedness of Solutions in Mathieu's Equation.**

Show that if $|\varepsilon|$ is small enough, then all solutions are bounded.

$$\ddot{x} + \left[1 + \varepsilon \sin 3t\right] x = 0.$$

[U. Utah PhD Preliminary Examination in Differential Equations, January 2004.]

27. [Oct. 30.] **Concrete Variational Equation.**

Let

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{pmatrix}.$$

Find the solution $\varphi(t, y) \in \mathbf{R}^3$ of

$$\frac{dx}{dt} = f\big(x(t)\big),$$
$$x(0) = y.$$

Find

$$\Phi(t, y) = D_2\, \varphi(t, y).$$

Show that it satisfies the variational equation

$$\frac{d\Phi}{dt} = Df\big(\varphi(t, y)\big) \cdot \Phi(t, y),$$
$$\Phi(0) = I.$$

[Perko, p. 84.]

28. [Nov. 1.] **Stability of the Origin in the Lorenz System**

The famous chaotic equations of meteorologist E. N. Lorenz model convective (predominantly vertical) flow realized by a fluid that is warmed from below and cooled from above. For b, r and σ positive constants,

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

(a) Show symmetry: if $(x(t), y(t), z(t))$ is a solution, so is $(-x(t), -y(t), z(t))$.

(b) The positive and negative z-axes are invariant sets.

(c) The origin is a critical point. If $0 < r < 1$ then the origin is a global attractor and the zero solution is asymptotically stable. [Hint: $V = x^2 + \sigma y^2 + \sigma z^2$.]

(d) If $r > 1$ the origin is unstable.

(e) The vector field is forward complete. There exists a compact positively invariant set (depending on b, r and σ) into which each forward trajectory enters but never leaves. [Hint: $V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$. Additional hints in W. Walter, *Ordinary Differential Equations,* Springer 1998, p. 330.]

29. [Nov. 3.] **Stationary Points of a Hamiltonian System.**

   Show that the system is Hamiltonian.

   $$\dot{x} = (x^2 - 1)(3y^2 - 1)$$
   $$\dot{y} = -2xy(y^2 - 1)$$

   Find the equilibrium points and classify them. Find the Hamiltonian. Using obvious exact solutions and the Hamiltonian property, draw a rough sketch of the phase diagram. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 79.]

30. [Nov. 6.] **Standard Proof of Differentiability of Solutions.**

   Suppose $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ is an open set, $f(t, x) \in \mathcal{C}^1(\Omega, \mathbf{R}^n)$ and $(s, p) \in \Omega$. Denote the solution of

   $$\dot{x} = f(t, x),$$
   $$x(s) = p,$$

   by $x(t, s, p)$, where t is any point in the domain of definition $\alpha(s, p) < t < \beta(s, t)$. In this problem we show that $x(t, s, p)$ is differentiable with respect to p and compute its differential.

   (a) Argue that $x(t, s, p)$ is defined and continuous in a neighborhood of t, $s = t_0$ and $p = x_0$.

   (b) Argue that the matrix function $Z(t, t_0, x_0)$ which solves

   $$\dot{Z}(t, t_0, x_0) = D_2 f(t, x(t, t_0, x_0))\, Z,$$
   $$Z(t_0, t_0, x_0) = I;$$

   is defined on the closed interval from t_0 to t.

   (c) Show that $x(t, s, p)$ is differentiable at (t, t_0, x_0) by showing that its differential equals $Z(t, t_0, x_0)$, namely, for $v \in \mathbf{R}^n$ show

   $$\lim_{h \to 0} \frac{1}{h} \left(x(t, t_0, x_0 + hv) - x(t, t_0, x_0) - Z(t, t_0, x_0)hv \right) = 0.$$

   [*cf.* James H. Liu, *A First Course in the Qualitative Theory of Differential Equations*, Prentice Hall 2003, p. 86.]

31. [Nov. 8.] **Soft Linearized Stability Existence Theorem.**

   Let A be a real $n \times n$ matrix all of whose eigenvalues satisfy $\Re \lambda < 0$. Suppose $f \in \mathcal{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ with $f(0) = 0$ and $Df(0) = 0$. Use the Implicit Function Theorem (Theorem 5.7 in text) to prove that there exists an $r > 0$ so that the integral equation

   $$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} f(y(s))\, ds$$

   has a solution $y \in \mathcal{C}_b([0, \infty), \mathbf{R}^n)$ for all $y_0 \in \mathbf{R}^n$ such that $|y_0| < r$. Compare to Theorem 3.11 of the text. [*cf.* Sideris, *Ordinary Differential Equations and Dynamical Systems*, Atlantis Press, 2013, p.86.]

32. [Nov. 10.] **Hartman-Grobman Theorem.**

Find a homeomorphism h in a neighborhood of 0 that establishes an topological conjugacy between the flow of the differential system and the flow of the linearized system, *i.e.*, $h(\psi(t,x)) = e^{tA}h(x)$ where $A = Df(0)$ and $\psi(t,x_0)$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$ and $\mathbf{x}(t_0) = x_0$, the nonlinear system given by

$$\dot{x} = -x$$
$$\dot{y} = -y + xz,$$
$$\dot{z} = z.$$

[In §4.2, Barreira & Valls discuss a proof, but you can guess h from the solutions and verify.]

33. [Nov. 13.] **Stable and Unstable Manifolds.**

Find the stable manifold W^s and unstable manifold W^u near the origin of the system

$$\dot{x} = -x$$
$$\dot{y} = -y + x^2$$
$$\dot{z} = z + y^2.$$

[*cf.* Perko, *Differential Equations and Dynamical Systems*, Springer, 1991, p. 116–117.]

34. [Nov. 15.] **Topological Conjugacy.**

Let

$$A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha > 0.$$

Explicitly construct the conjugacy found in the proof of Theorem 7.3 of Sideris's text.

[*cf.,* Gerald Teschl, *Ordinary Differential Equations and Dynamical Systems,* Amer. Math. Soc., 2012, p.270.]

35. [Nov. 17.] **Center Manifold.**

Find a center manifold for the system

$$\dot{x} = -xy$$
$$\dot{y} = -y + x^2 - 2y^2$$

through the rest point at the origin. Find a differential equation for the dynamics on the center manifold. Show that every nearby solution is attracted to the center manifold. Determine the stability of the origin.

Hint: Look for a center manifold that is a graph $y = \psi(x)$ of the form

$$\psi(x) = \sum_{k=2}^{\infty} a_k x^k$$

using the condition of invariance $\dot{y} = \psi'(x)\dot{x}$ and $\psi(0) = \psi'(0) = 0$. Find the first few terms of the expansion, guess the rest and check. Then get the equation for the induced flow on the center manifold. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 304.]

36. [Nov. 20.] **Periodic Orbit in Predator-Prey System.**

A generalization of a predator-prey system given by Brauer and Castillo-Chavez is

$$\dot{x} = x\left(1 - \frac{x}{30} - \frac{y}{x+10}\right),$$
$$\dot{y} = y\left(\frac{x}{x+10} - \frac{1}{3}\right).$$

(a) Show that the fixed points are $(0,0)$, $(30,0)$ and $(5,12.5)$ and have saddle, saddle, source type, resp.

(b) Show that the region bounded by $0 \le x$, $0 \le y$ and $x + y \le 50$ is forward invariant.

(c) Show that there is no orbit Γ with $\alpha(\Gamma) = \{(30,0)\}$ and $\omega(\Gamma) = \{(0,0)\}$. Conclude that there is a nonconstant periodic orbit.

[R. C. Robinson, *An Introduction to Dynamical Systems Continuous and Discrete*, Pearson/Prentice Hall, 2004, p. 238.]

37. [Nov. 22.] **Dulac's Criterion.**

Prove the following theorem of Dulac.
Theorem. Let $A \subset \mathbb{R}^2$ be an annular domain. Let $f \in \mathcal{C}^1(A, \mathbb{R}^2)$ and let $\rho \in \mathcal{C}^1(A, \mathbb{R})$. If $\text{div}(\rho f)$ is not identically zero and does not change signs in any open subset of A then the equation $x' = f(x)$ has at most one periodic solution in A.

Use this to show that the van der Pol oscillator ($\lambda = \text{const.} \ne 0$)

$$\dot{x} = y$$
$$\dot{y} = -x + \lambda(1 - x^2)y$$

has at most one limit cycle in the plane. Hint: let $\rho = (x^2 + y^2 - 1)^{-1/2}$. [Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 90.]

38. [Nov 24.] **Brusselator System.**

Show that there is a nonconstant periodic trajectory for the system

$$\dot{x} = 1 - 4x + x^2 y$$
$$\dot{y} = 3x - x^2 y$$

[University of Utah Ph.D. Preliminary Examination in Differential Equations, August 2004.]

39. [Nov. 27.] **Stability of a Periodic Orbit.**

Find a periodic solution to the system

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$
$$\dot{z} = -z,$$

Let Σ be a halfplane whose boundary is the z axis. Determine the Poincare map $P : \Sigma \to \Sigma$ and determine its differential at the periodic orbit. Determine the stability type. In particular, compute the Floquet Multipliers for the fundamental matrix associated with the periodic orbit. Is it orbitally asymptotically stable? Is it asymptotically stable? [*cf.* Perko, Differential Equations and Dynamical Systems, Springer, 1991, p. 201.]

40. [Nov. 29.] **Stability in a Non-Autonomous Equation.** Show the stability of the solution

$$x(t) = \sqrt{2b} \cos \frac{1}{2} t$$

of the equation

$$\ddot{x} + \left(\frac{1}{4} - 2\epsilon b \cos^2 \frac{1}{2} t \right) x + \epsilon x^3 = 0.$$

[D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, pp. 334.]

41. [Dec. 1.] **Perturbation of Stable Solution of van der Pol Equation.**

Suppose $g \in C^2(\mathbb{R}^2)$ and $\mu > 0$. Show that there is an $\varepsilon_0 > 0$ such that for all ε satisfying $|\varepsilon| < \varepsilon_0$ there is a unique periodic solution of the perturbed van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \varepsilon g(x, \dot{x})$$

in the neighborhood of the unique nontrivial periodic solution of

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

Can you prove that this solution is orbitally asymptotically stable? [J. Hale, *Ordinary Differential Equations,* Dover 2009; repub. R. Krieger 1980; orig. J. Wiley 1969, p. 227.]

42. [Dec. 4.] **Perihelion of Mercury.**

The orbital equation of a planet about the sun is

$$\frac{d^2 u}{d \vartheta^2} + u = k(1 + \varepsilon u^2)$$

where $u = \frac{1}{r}$ and r, ϑ are polar coordinates, $k > 0$ is a celestial constant and $k\varepsilon u^2$ is a relativistic correction term. Obtain a perturbation solution with initial condition $u(0) = k(e + 1)$, $\dot{u}(0) = 0$ where e is the eccentricity of the unperturbed orbit. These are the conditions at the perihelion, the nearest point to the sun on the unperturbed orbit. Show that the expansion to order ε predicts that in each orbit, the perihelion advances by $2k^2\pi\varepsilon$. [D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers, 4th ed.*, Oxford U. Press, 2007, p. 181.]

43. [Dec. 6.] **Period of the van der Pol Oscillator.**

For $0 < \varepsilon$ small enough, we showed that there is initial data $x_0(\varepsilon)$ such that the solution $x(t, x_0(\varepsilon), \varepsilon)$ of the van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0,$$
$$x(0) = x_0(\varepsilon), \quad \dot{x}(0) = 0.$$

is a nonconstant periodic orbit of period $T(\varepsilon)$. Find the Taylor expansion of $T(\varepsilon)$ up to second order.

[Chicone, *Ordinary Differential Equations with Applications,* Springer 1999, p. 324.]

The last day to turn in any remaining homework is Tuesday, Dec. 12.