Geometric Analysis 1: Geometry of Surfaces

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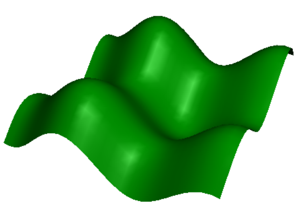
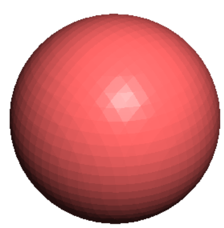
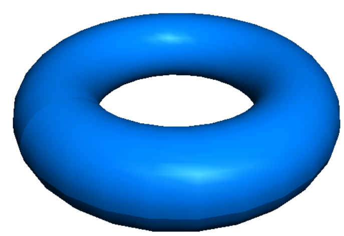
Geomoetric analysis uses PDE's and complex analysis to say something about geomoetry. These notes cover basic ideas of curves and surfaces.
We can think of a surface M^2 as a subset of R^3. Some examples of surfaces are:

- Graphs of functions:
$$\mathbb{G}^2 = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), (x, y) \in U\}$$
where $U \subseteq \mathbb{R}^2$ is open.

- Level sets:
$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$
. This is the standard unit sphere.

- Parameterized surfaces:
$$\mathbb{T}^2 = \{((A + a\cos\varphi)\cos\theta, (A + a\cos\varphi)\sin\theta, a\sin\varphi) | a < A \in \mathbb{R}, \varphi, \theta \in \mathbb{R}\}$$
is a torus with radii $a < A$.

A surface can locally be given by a curvilinear coordinate chart, also called a parameterization. Let $U \subset \mathbb{R}^2$ be open. Let $X : U \longrightarrow \mathbb{R}^3$ be a C^1 function. Then we want $M = X(U)$ to be a surface. At each point $p \in X(U)$ we can identify tangent vectors u_1 and u_2 to the surface. If $p = X(a)$ for some $a \in U$, then we define

$$X_i(a) = \frac{\partial X}{\partial u_i}(a)$$

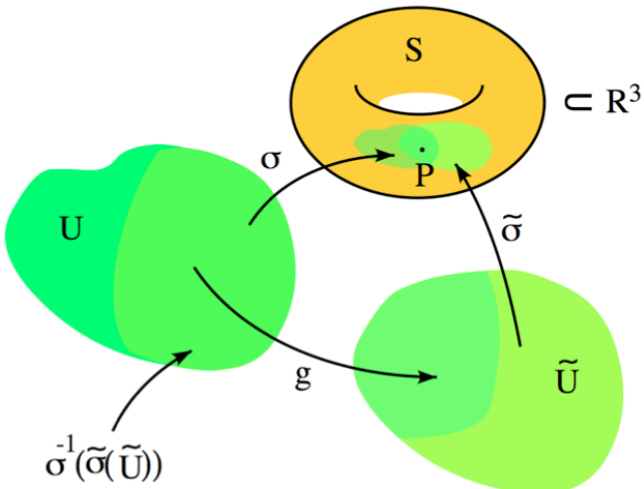
which gives us two vectors in $\mathbb{R}^3$ tangent to the coordinate curves. To avoid singularities at p, we assume that all $X_1(p)$ and $X_2(p)$ are linearly independent vectors. Then the tangent plane to the surface at p is

$$T_pM = \text{span}\{X_1(p), X_2(p)\}$$

Definition. A connected set $M \subset \mathbb{R}^3$ is a regular surface if for each $p \in M$ there is a neighborhood $V \subset \mathbb{R}^3$ and map $X : U \longrightarrow V \cap M$ of an open set $U \in \mathbb{R}^2$ onto $V \cap M$ such that

1. $X : U \longrightarrow V \cap M$ is a smooth homeomorphism.
2. $X_1(U)$ and $X_2(U)$ are linearly independent for all $u \in U$.

The function X is called a chart and can be used to establish a local coordinate system on the manifold.



Suppose $S \subset \mathbb{R}^3$ is a surface and at $p \in S$ there are two charts $\varphi : U \longrightarrow S$ and $\psi : V \longrightarrow S$ such that U and V are open subsets of $\mathbb{R}^2$ and $p \in \varphi(U) \cap \psi(V)$. Let $u \in \varphi^{-1}(\psi(V))$. We can define the transition function $v = \psi^{-1}(\varphi(u)$, which gives the change of coordinates map. The Inverse Function Theorem guarantees that these maps are smooth and invertible.

To talk about surfaces, it is important to consider when two surfaces are essentially the same. Different kinds of equivalence are extablished by different kinds of functions between surfaces.

Definition. Two surfaces M and N are *diffeomorphic* if there is a function $f : M \longrightarrow N$ such that f is invertible and both f and f^{-1} are smooth.

To abstract the idea of a regular surface, we drop the requirement that $M \subset \mathbb{R}^3$. We define a differentiable manifold as a topological space M that has a collection of coordinate charts whose transition functions are smooth and consistently defined. The atlas of charts with corresponding transition functions is called a differential structure.

The question naturally arises of whether differentiable manifolds exist that do not arise as submanifolds of Euclidean space. Whitney's embedding theorem answers this question in the negative, as long as we allow larger codimensions.

Whitney's Embedding Theorem. *Let M^n be a differentiable manifold of dimension n and $N \geq 2n + 1$. Then M^n is diffeomorphic to some $W^n \subset \mathbb{R}^N$, an embedded regular submanifold.*

The Euclidean structure of $\mathbb{R}^3$, the usual dot product, gives a way to measure lengths and angles of vectors. If $V = (v_1, v_2, v_3)$, then its length

$$|V| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{V \cdot V}$$

If $W = (w_1, w_2, w_3)$, then the angle $\alpha = \angle(V, W)$ is given by

$$\cos \alpha = \frac{V \cdot W}{|V||W|}$$

If $\gamma : [a, b] \longrightarrow M \subset \mathbb{R}^3$ is a continuously differentiable curve, its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}| dt$$

If the curve is confined to a coordinate patch $\gamma([a, b]) \subset X(U) \subset M$, then we may factor through the coordinate chart. There are continuously differentiable $u(t) = (u_1(t), u_2(t)) \in U$ so that $\gamma(t) = X(u_1(t), u_2(t))$ for all $t \in [a, b]$. Then the tangent vector my be written

$$\dot{\gamma}(t) = X_1(u_1(t), u_2(t))\dot{u}_1(t) + X_2(u_1(t), u_2(t))\dot{u}_2(t)$$

so its length is

$$|\dot{\gamma}|^2 = X_1 \cdot X_1 \dot{u}_1^2 + 2X_1 \cdot X_2 \dot{u}_1 \dot{u}_2 + X_2 \cdot X_2 \dot{u}_2^2$$

For $i, j = 1, 2$ the Riemannian metric is given my the matrix function $g_{ij}(u) = X_i(u) \cdot X_j(u)$. Evidently, $g_{ij}(u)$ is smoothly varying, symmetric, and positive definite. Thus

$$|\dot{\gamma}(t)|^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(u(t))\dot{u}_i(t)\dot{u}_j(t)$$

The length of the curve on the surface is determined by its velocity in the coordinate patch $\dot{u}(t)$ and the metric $g_{ij}(u)$.

A vector field on the surface is also determined by functions in U using the basis. Thus if V and W are tangent vector fields, they may be written

$$V(u) = v^1(u)X_1(u) + v^2(u)X_2(u)$$

and

$$W(u) = w^1(u)X_1(u) + w^2(u)X_2(u)$$

The $\mathbb{R}^3$ dot product can also be expressed by the metric:

$$V \cdot W = \langle V, W \rangle = \sum_{i,j=1}^{2} g_{ij} v^i w^j$$

where $\langle \cdot, \cdot \rangle$ is an inner product on T_pM that varies smoothly on M. This Riemannian metric is also called the First Fundamental Form.

If V and W are nonvanishing vector fields on M then their angle $\alpha = \angle(V, W)$ satisfies

$$\cos\alpha = \frac{\langle V, W\rangle}{|V||W|}$$

which depends on the coordinates of the vector fields and the metric. If $D \subset U$ is a piecewise smooth subdomain in the patch, the area if $X(D) \subset M$ is also determined by the metric

$$A(X(D)) = \int_D |X_1 \times X_2| du_1 du_2 = \int_D \sqrt{\det(g_{ij}(u))} du_1 du_2$$

since if $\beta = \angle(X_1, X_2)$, then

$$\begin{aligned} |X_1 \times X_2|^2 &= \sin^2\beta |X_1|^2 |X_2|^2 \\ &= (1 - \cos^2\beta)|X_1|^2|X_2|^2 \\ &= |X_1|^2|X_2|^2 - (X_1 \cdot X_2)^2 \\ &= g_{11}g_{22} - g_{12}^2 \end{aligned}$$

For the graph $\mathbb{G}^2 = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y) \text{ and } (x, y) \in U\}$ take the patch $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$. The metric components are $g_{ij} = X_i \cdot X_j$ so

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}$$

where $f_i = \frac{\partial f}{\partial u_i}$. This gives the usual formula for area

$$\det(g_{ij}) = 1 + f_1^2 + f_2^2$$

so

$$A(X(D)) = \int_D \sqrt{1 + f_1^2 + f_2^2} du_1 du_2$$

If we endow an abstract differentiable manifold M^n with a Riemannian Metric, a smoothly varying inner product on each tangent space that is consistently defined on overlapping coordinate patches, the resulting object is a Riemannian Manifold.

Now that we have a metrics on manifolds, we can define equivalence between manifolds in a narrower way. Two manifolds are *isometric* if there is a distance-preserving map between them.

As before, the question arises of whether there are Riemannian manifolds that do not arise as submanifolds of Euclidean space with the induced differential structure and Riemannian metric. Provided we allow sufficiently large codimension, such manifolds do not exist, as shown by Nash.

Nash's Isometric Immersion Theorem. *Let M^n be a smooth Riemannian manifold of dimension n and $N \ge n^2 + 10n + 3$. Then M^n is isometric to a smooth immersed manifold $W^n \subset \mathbb{R}^N$ with induced Riemannian metric.*