Calculus of Variations: The Direct Approach

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June 7, 2010

The original lecture slides are available online at:

`http://www.math.utah.edu/~treiberg/DirectMethodSlides.pdf`

An Example of a Variational Problem: Curve of Shortest Length

Suppose we wish to find the shortest curve from (a, y_1) to (b, y_2) in the Euclidean plane (we of course already know how to do this, but simply wish to introduce rigorous notation for future examples).

We then look for a curve $\gamma(t) = (t, u(t))$ with endpoints as described above, or in other words, a $u(t)$ belonging to the admissible set

$$\mathcal{A} = \{w \in \mathcal{C}^1([a, b]) : w(a) = y_1,\ w(b) = y_2\}$$

It must minimize the length integral, which is written

$$\mathrm{L}(u) = \int_a^b \sqrt{1 + \dot{u}^2(t)}\, dt.$$

The minimizing curve must satisfy the Euler equation, which is in this case

$$\frac{d}{dt}\left(\frac{\dot{u}}{\sqrt{1 + \dot{u}^2}}\right) = 0$$

and the solution is a straight line $u(t) = c_1 t + c_2$, as we expected.

Let us now add an additional constraint to the same problem - that the area under the curve be a fixed number j_0. Our admissible set is now

$$A' = \left\{ w \in \mathcal{C}^1 : \begin{array}{l} w(a) = y_1, \\ w(b) = y_2, \\ \int_a^b u(t)dt = j_0 \end{array} \right\}$$

This is called the Isoperimetric Problem. As shown in the proof outline below, we will see that the associated Euler-Lagrange equations demand the minimum-length curve have constant curvature, thus must be an arc of a circle. Note that in some cases there may be no minimum.

We may solve a more general problem by letting $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary (to act as the interval $[a,b]$ from before) and letting $\phi \in \mathcal{C}^1(\overline{\Omega})$ be the boundary conditions. We then look for functions $u(t)$ that maximize some

$$I(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

subject to the constraint that

$$J(u) = \int_\Omega g(x, u(x), Du(x))\, dx = j_0$$

In our previous problem, $I(u)$ was the negative of arc length and $J(u)$ was the area under the curve $u(t)$.

Look at two-parameter variations $U(x, \varepsilon_1, \varepsilon_2) = u(x) + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x)$, where $\eta_1(z) = \eta_2(z) = 0$ are smooth functions which vanish on the boundary of Ω. Thus U is in our admissable set. Define the following:

$$I(\varepsilon_1, \varepsilon_2) = \int_a^b f(t, U, DU)\, dx$$
$$J(\varepsilon_1, \varepsilon_2) = \int_a^b g(t, U, DU)\, dx$$

Note that I is at a maximum when $\varepsilon_1 = \varepsilon_2 = 0$

Using Lagrange Multipliers, there is a constant λ so that the solution is the critical point of the Lagrange function

$$\mathcal{L}(\varepsilon_1, \varepsilon_2) = I(\varepsilon_1, \varepsilon_2) + \lambda J(\varepsilon_1, \varepsilon_2) = \int_a^b h(t, U, DU)\, dt$$

where

$$h(t, U, DU) = f(t, U, DU) + \lambda g(t, U, DU).$$

We may use this to find the weak Euler-Lagrange Equations

$$\left.\frac{\partial \mathcal{L}}{\partial \varepsilon_i}\right|_{\varepsilon_1 = \varepsilon_2 = 0} = \int_a^b \left\{ \frac{\partial h}{\partial u} \eta_i + D_p h \bullet D_x \eta_i \right\} dx = 0,$$

Integrating by parts (which we assume is OK) yields

$$\int_a^b \eta_i \left\{ \frac{\partial h}{\partial u} - \operatorname{div}(D_p h) \right\} dt = 0.$$

And remembering that η_i are arbitrary, this can only hold if

$$\frac{\partial h}{\partial u} - \operatorname{div}(D_p h) = 0.$$

Returning to the original isometric problem we were interested in and plugging in values, we have

$$h = f + \lambda g = \sqrt{1 + \dot{u}^2} + \lambda u.$$

Inserting this into the Euler-Lagrange Equation and with some algebraic manipulation, we get that $0 = \lambda - \kappa$, or in other words, that the curvature κ is a constant not dependant on t. This means our solution is an arc of a circle.

Dirichlet's Principle and Hilbert

Dirichlet's Principle is derived from an electrostatics problem: If two electric battery poles are attached at points to a thin conducting sheet, the potential across the sheet is the solution of a boundary value problem, and one can look for a solution which produces minimal heat. Generalizing,

Dirichlet's Principle. Let $G \subset \mathbf{R}^2$ (or in a smooth surface) be a compact domain and $\phi \in \mathcal{C}(\partial G)$. Then there is $u \in \mathcal{C}^1(G) \cap \mathcal{C}(\overline{G})$ that satisfies $u = \phi$ on ∂G and minimizes the Dirichlet Integral

$$D[u] = \int_G |Du|^2 \, dA.$$

Moreover, $\Delta u = 0$ on G.

Dirichlet was mistaken in assuming a minimizer must exist. Weierstrass found the flaw and Hilbert finally proved the principle rigorously (but assuming appropriate smoothness).

Hilbert later (in his list of famous problems to attack in the 20th century) suggested the following avenues be explored in Calculus of Variations: Are solutions of regular variation problems always analytic? Does a solution always exist, and if not, when are we sure that a solution must exist? Also, can we always modify the definition of 'solution' in a meaningful way for any problem?

Due to these questions, much progress was made in the 20th century.

The Direct Method

The direct method for solution to minimization problem on a functional $\mathcal{F}(u)$ is as follows:

Step 1: Find a sequence of functions such that $\mathcal{F}(u_n) \to inf_{\mathcal{A}}\mathcal{F}(u)$

Step 2: Choose a convergent subsequence $u_{n'}$ which converges to some limit u_0. This is the candidate for the minimizer.

Step 3: Exchange Limits:

$$\mathcal{F}(u_0) = \mathcal{F}\left(\lim_{n' \to \infty} u_{n'}\right) = \lim_{n' \to \infty} \mathcal{F}(u_{n'}) = \mathcal{I}.$$

There are obviously issues with assuming some of the above steps are possible, for example:

1) There may not be a lower bound.
2) The set A of admissible functions may not be compact.
3) Only allowed to exchange limits if $\mathcal{F}$ is lower-semicontinuous

With 'nice' enough spaces and functions, though, the direct method assures existence of a minimizing solution.

Illustration with Poisson Minimization Problem

The Poisson minimization problem uses the following functional:

$$\mathcal{F}(u) = \int_\Omega \tfrac{1}{2}|Du|^2 + \psi u \, dx.$$

Here, $\psi \in \mathcal{L}^2(\Omega)$ and $\phi \in \mathcal{C}^1(\overline{\Omega})$.

The Euler Equation is

$$0 = \psi - \operatorname{div}(Du)$$

This is usually written $u = \phi$ on $\partial\Omega$, $\Delta u = \psi$ in Ω and is Poisson's Equation.

We will see using the direct method that the following theorem holds true:

Poisson's Minimization Problem.

Let $\Omega \subset \mathbf{R}^n$ be a bounded, connected domain with smooth boundary. Let $\phi, \psi \in \mathcal{C}^\infty(\overline{\Omega})$. For $u \in C^1(\Omega)$, let

$$\mathcal{F}(u) = \int_\Omega \tfrac{1}{2}|Du|^2 + \psi u \, dx.$$

Then there is a unique $u_0 \in \mathcal{C}^\infty(\overline{\Omega})$ with $u_0 = \phi$ on $\partial\Omega$ such that

$$\mathcal{F}(u_0) = \inf_{u \in \mathcal{A}'} \mathcal{F}(u)$$

where $\mathcal{A}' = \left\{u \in C(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) : u = \phi \text{ on } \partial\Omega.)\right\}$. Also, $\Delta u_0 = \psi$ in Ω.

Note: We enlarge the space of admissible functions using the Hilbert Space:

$$\mathcal{H}^1(\Omega) := \left\{ u \in \mathcal{L}^2(\Omega) : \begin{array}{l} \text{all distributional derivatives} \\ \text{exist and } \frac{\partial u}{\partial x_i} \in \mathcal{L}^2(\Omega) \text{ for all} \\ i. \end{array} \right\}$$

This makes sense because to prove an inequality, you only need to prove it on a dense set, and $\mathcal{H}^1(\Omega)$ is the completion of $\mathcal{C}^\infty(\overline{\Omega})$. Similarly denote by $\mathcal{H}^1_0(\Omega)$ the completion of $C_0^\infty(\overline{\Omega})$. Denote by $\mathcal{A}^1$ the extension of $\mathcal{A}'$ to this Hilbert space, as follows:

$$\mathcal{A}^1 := \left\{u \in \mathcal{H}^1(\Omega) : u - \phi \in \mathcal{H}^1_0(\Omega)\right\}.$$

Coercivity. We need to address the three issues listed above to make sure the direct method is going to work. Our first task is to prove $\mathcal{F}$ is bounded below (coercive).

Lemma: There are constants $c_1, c_2 > 0$ depending on ψ and Ω so that for all $u \in \mathcal{A}^1$,

$$\mathcal{F}(u) \geq c_1 \|u\|^2_{\mathcal{H}^1} - c_2.$$

It follows easily that $\mathcal{F}$ is bounded below by $-c_2$ and

$$\mathcal{I} = \inf_{v \in \mathcal{A}^1} \mathcal{F}(v)$$

exists and is finite. The proof is a bit involved and is not included here. We now know that we may choose a minimizing sequence $u_n \in \mathcal{A}^1$ so that

$$\lim_{n \to \infty} \mathcal{F}(u_n) = \mathcal{I}.$$

Compactness. We can assume $\mathcal{F}(u_n) < \mathcal{I} + 1$ for all n by renumbering, so

$$\|u_n\|_{\mathcal{H}^1} \leq \frac{\mathcal{I} + 1 + c_2}{c_1}.$$

FACT: In any Hilbert Space, *e.g.* in $\mathcal{H}^1$, any bounded sequence $\{u_n\}$ is weakly sequentially compact: there is a subsequence $\{u_{n'}\}$ that weakly converges in $\mathcal{H}^1$ to $u_0 \in \mathcal{H}^1$. That is, for any $v \in \mathcal{H}^1$,

$$\langle u_{n'}, v \rangle_{\mathcal{H}^1} \to \langle u_0, v \rangle_{\mathcal{H}^1} \qquad \text{as } n' \to \infty.$$

FACT: The embedding $\mathcal{H}^1(\Omega) \subset \mathcal{L}^2(\Omega)$ is compact. *i.e.*, by going to a sub-subsequence if necessary, we may assume $u_{n''} \to u_0$ in $\mathcal{L}^2(\Omega)$.

FACT: $\mathcal{A}^1$ is a closed subspace of $\mathcal{H}^1(\Omega)$. If all $u_{n'}$ belong to a closed subspace and $\{u_{n'}\}$ converges weakly to u_0 in $\mathcal{H}^1$, then u_0 also belongs to the closed subspace. *i.e.*, $u_0 \in \mathcal{A}^1$.

u_0 is the candidate to be the minimizer of the variational problem.

Lower Semi-Continuity. We need this in order to switch the order of $\mathcal{F}$ and the limit in Step 3 of the Direct Method.

Lemma: Let u_n be a minimizing sequence for $\mathcal{F}(u)$ such that $u_n \to u_0$ strongly in $\mathcal{L}^2(\Omega)$ and weakly in $\mathcal{H}^1(\Omega)$. Then

$$\mathcal{F}(u_0) \leq \liminf_{n \to \infty} \mathcal{F}(u_n).$$

Proof. Since $u_n \to u_0$ in $\mathcal{L}^2(\Omega)$, $\int_\Omega \psi u_n \to \int_\Omega \psi u_0$ and $\|u_n\|_{\mathcal{L}^2} \to \|u_0\|_{\mathcal{L}^2}$.

In any Hilbert Space the norm is SWLSC: $\|u_0\|_{\mathcal{H}^1} \le \liminf_{n\to\infty} \|u_n\|_{\mathcal{H}^1}$.

$$\begin{aligned}\mathcal{F}(u_0) &= \tfrac{1}{2}\|Du_0\|^2_{\mathcal{L}^2} + \int \psi u_0 = \tfrac{1}{2}\|u_0\|^2_{\mathcal{H}^1} - \tfrac{1}{2}\|u_0\|^2_{\mathcal{L}^2} + \int \psi u_0 \\ &\le \liminf_{n\to\infty} \left\{ \tfrac{1}{2}\|u_n\|^2_{\mathcal{H}^1} - \tfrac{1}{2}\|u_n\|^2_{\mathcal{L}^2} + \int \psi u_n \right\} = \liminf_{n\to\infty} \mathcal{F}(u_n) = \mathcal{I}.\end{aligned}$$

Uniqueness of Solution. Uniqueness follows from the convexity of the functional $\mathcal{F}(u)$, which we must define.

$\mathcal{F} \in \mathcal{C}^1$ is *convex* on $\mathcal{A}^1 \subset \mathcal{H}^1$ if

$$\mathcal{F}(u+w) - \mathcal{F}(u) \ge D\mathcal{F}(u)[w] \text{ whenever } u, u+w \in \mathcal{A}^1.$$

$\mathcal{F}$ is *strictly convex* if "=" holds iff $w = 0$.

We have not yet proven $\mathcal{F} \in \mathcal{C}^1$ so this is the first step in the argument. By definition, this is true if $\mathcal{F}$ is differentiable and $D\mathcal{F} : \mathcal{H}^1 \to (\mathcal{H}^1)^*$ is continuous. We prove them here.

$\mathcal{F}$ is differentiable:

$|\mathcal{F}(u+v) - \mathcal{F}(u) - D\mathcal{F}(u)[v]| = \frac{1}{2}\int_\Omega |Dv|^2\,dx \le \frac{1}{2}\|v\|^2_{\mathcal{H}^1} = \mathbf{o}\left(\|v\|_{\mathcal{H}^1}\right)$ as $\|v\|_{\mathcal{H}^1} \to 0$.

$D\mathcal{F}$ is continuous: For $u, v, w \in \mathcal{H}^1$,

$$\begin{aligned}|D\mathcal{F}(u)[w] - D\mathcal{F}(v)[w]| &= \left| \int_\Omega (Du - Dv) \cdot Dw\,dx \right| \\ &\le \|D(u-v)\|_{\mathcal{L}^2}\|Dw\|_{\mathcal{L}^2} \le \|u-v\|_{\mathcal{H}^1}\|w\|_{\mathcal{H}^1}.\end{aligned}$$

Thus

$$\|D\mathcal{F}(u) - D\mathcal{F}(v)\|_{(\mathcal{H}^1)^*} = \sup_{w\ne 0} \frac{|D\mathcal{F}(u)[w] - D\mathcal{F}(v)[w]|}{\|w\|_{\mathcal{H}^1}} \to 0$$

as $\|u-v\|_{\mathcal{H}^1} \to 0$.

It is easily checked that our particular $\mathcal{F}$ in the Poisson problem is convex, so it follows with some work that the solution is unique.

What have we shown?

Using the Direct Method, we showed the existence of a unique weak solution $u_0 \in \mathcal{H}^1$ of the Dirichlet problem for Poisson's Equation. If the coefficients are known to be smoother (smooth boundary, smooth ψ and ϕ) then the solution u will also have more regularity.