Penrose Tiling

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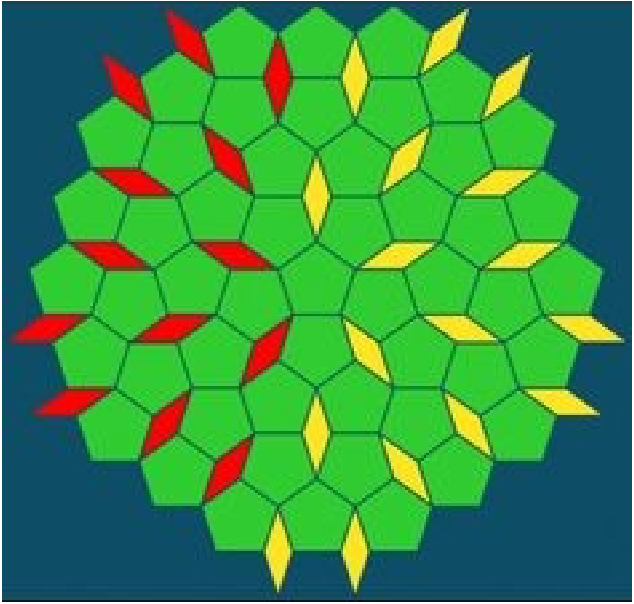
The URL for these Beamer Slides: *"Penrose Tiling"*

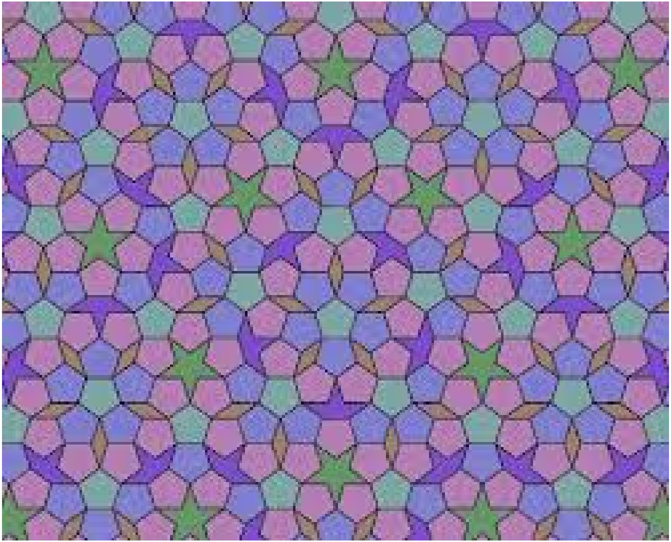
`http://www.math.utah.edu/~treiberg/PenroseSlides.pdf`

- David Austin, Penrose Tiles Talk Across Miles, American Mathmatical Society Feature Column: Monthly essays on mathematical topics; Penrose Tilings Tied up in Ribbons.
- N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings of the plane I, II, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen Series A, 84 (1), March, 1981, 39–66.
- Martin Gardner, *Penrose Tiles to Trapdoor Ciphers...and the return of Dr. Matrix,* American Mathematical Association, Washington D. C., 1997;orig. publ. W. H. Freeman and Co., New York, 1989, 1–30.
- Branko Grünbaum and G. C. Shephard, *Tilings and Patterns, 2nd ed.,* Dover, Mineola, 2016; orig. publ. W. H. Freeman and Co., New York, 1987, 519–582.

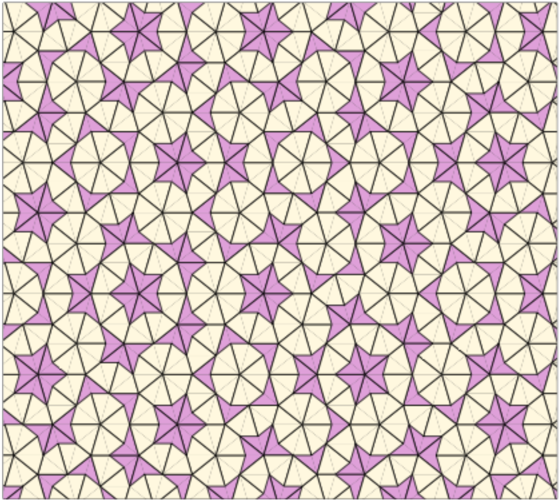
- Penrose tilings
  - Periodic tilings
  - Examples of non-periodic tilings
- Penrose's rhombic prototiles
  - Matching rules
  - Quasiperiodic tilings in Islamic architecture
  - Deflation
  - Existence of a tiling and a program to draw it
  - Non-periodicity of this tiling
  - Kites and darts
  - Inflation and aperiodicity
- de Bruijn's up-down generation
  - Uncountably many Penrose tilings
- de Bruijn's Pentagrid method
  - Index of a vertex in a rhombic tiling
  - Wieringa roof
- Tiling is projection of center points of cubes from the cubic lattice that touch a plane in $\mathbf{R}^5$

The pentagon does not tile the plane. Roger Penrose, who is a famous cosmologist, wondered how close can you come to filling the plane with pentagons. In answering his question, he came up with three sets of prototiles with the property that any tiling of the plane using just these prototiles is always non-periodic. In fact, his tilings have remarkable self symmetry whose diffraction pattern has five-fold symmetry, suggesting that they are models for quasicrystals.





8. Penrose's second aperiodic tiling by kites and darts.



9. Penrose's third aperiodic tiling by thick and thin rhombs.

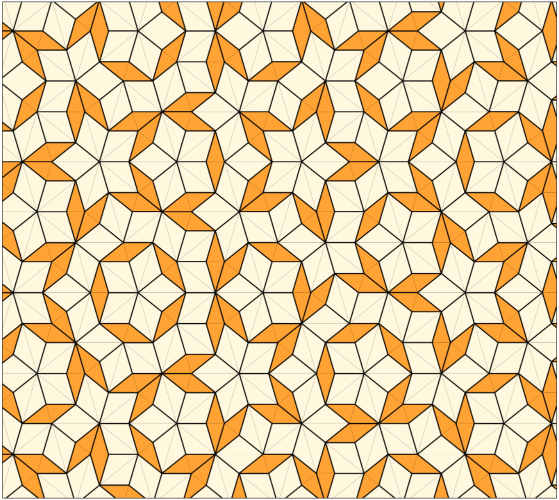
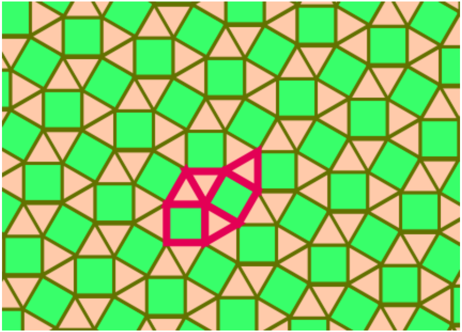
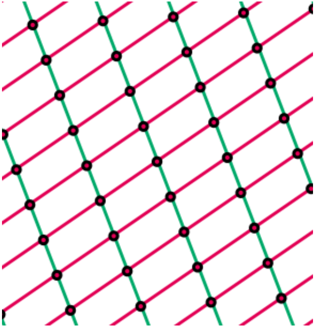


Figure: Roger Penrose on tiles

In 1957, Penrose finished his PhD at Cambridge under Hodge and Todd. He worked at Birbeck College London and eventually moved to Oxford in 1973. He was knighted in 1994. He developed topological methods in cosmology, invented twistors and collaborated with Hawking on gravitational collapse. He discovered his nonperiodic tiling in 1974 when trying to find the densest packing of the plane by regular pentagons.

Penrose saw M. C. Escher's art at a conference and sent him a picture of a nonending staircase. Escher used the tiling in one of his pattern paintings.





In 1619, Johannes Kepler created the first list of all the Archimedean Tilings, tilings that can be created from sets of regular polygons. Here is an example using just squares and triangles. The fundamental region is indicated in red. The fundamental region is translated by the vectors of the lattice whose coordinates are

$$\mathbf{w} = k\mathbf{v}_1 + \ell\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent vectors and k and ℓ are integers.

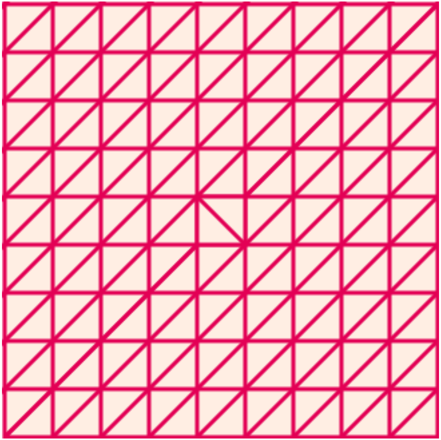


Figure: Nonperiodic tiling by triangles

One can tile the plane by right isosceles triangles simply by subdividing the sqares in the standard lattice by SW to NE diagonals. Taking the other diagonal in one square breaks the translation invariance and produces a nonperiodic tiling.

13. Ulam's nonperiodic tiling

65	64	63	62	61	60	59	58	57
66	37	36	35	34	33	32	31	56
67	38	17	16	15	14	13	30	55
68	39	18	5	4	3	12	29	54
69	40	19	6	1	2	11	28	53
70	41	20	7	8	9	10	27	52
71	42	21	22	23	24	25	26	51
72	43	44	45	46	47	48	49	50
73	74	75	76	77	78	79	80	81

Figure: Spiraling numbers

Ulam used the same right triangular tiles. Start by numbering the squares in the grid by spiraling about the center square. Color the prime squares red.

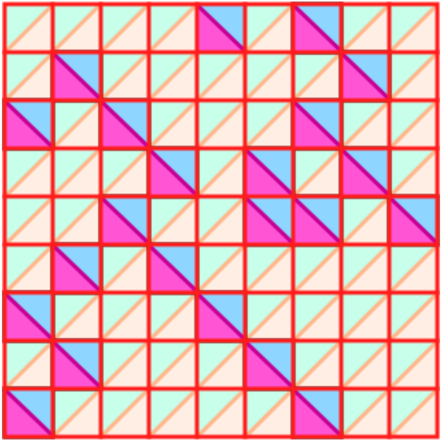


Figure: Ulam's nonperiodic tiling.

Ulam put the diagonals in SW to NE in the white squares and the other way in the red. The result is another nonperiodic tiling.

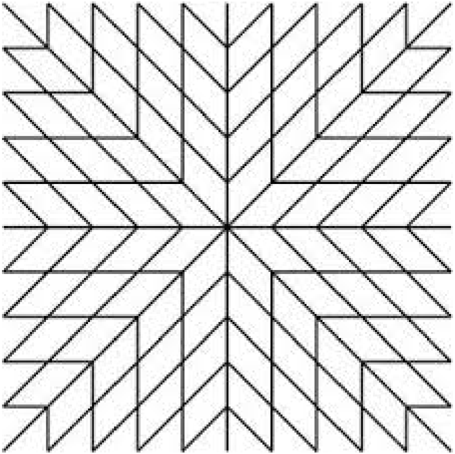
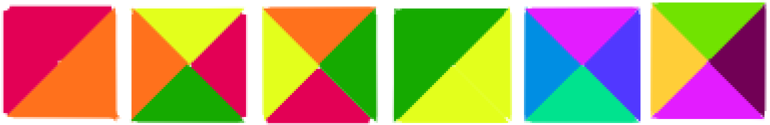


Figure: Nonperiodic star tiling by rhombuses

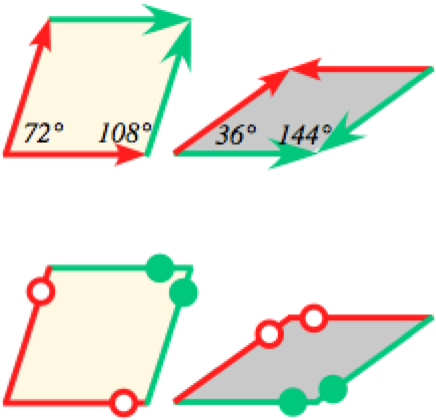
Here is a way to make a nonperiodic tiling with rhombus tiles.

As we have seen, the triangle and rhombuses may tile the plane periodically or nonperiodically. The remarkable discovery in the last fifty years is that there are tiles that admit infinitely many tilings of the plane, yet no such tiling is periodic. A set of tiles with this property are called aperiodic.

In 1961, Hao Wang studied tiling by unit squares whose edges were colored. Wang tiles must be placed edge-to-edge; colors on contiguous edges must match and only translations (no rotations or reflections) of the prototiles are allowed. Wang's problem was to find a procedure to decide whether a given set of prototiles will tile just by placing tiles so that abutting edges are of the same color. This question relates symbolic logic and the study of Turing machines. Wang conjectured that if the tiles can tile the plane, then they can do so periodically.



Robert Berger's 1966 Harvard thesis in applied mathematics showed that Wang's conjecture is false: there is a set of 20,426 Wang prototiles that tiles only nonperiodically.

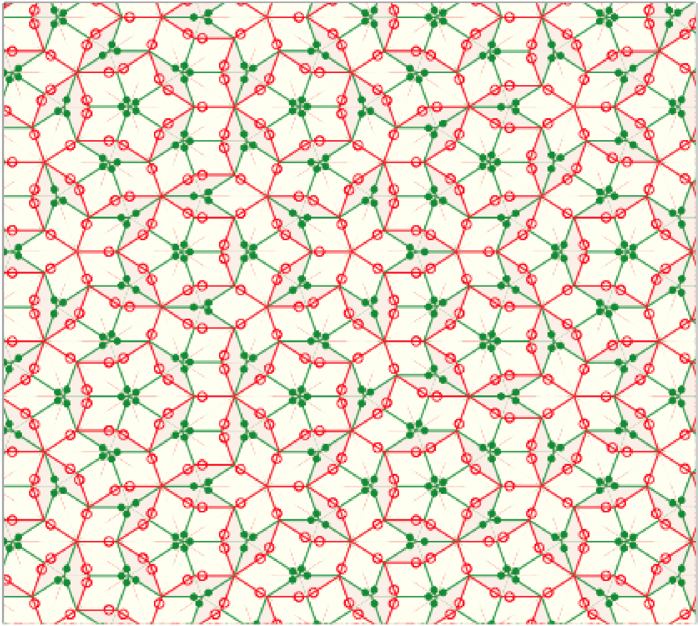


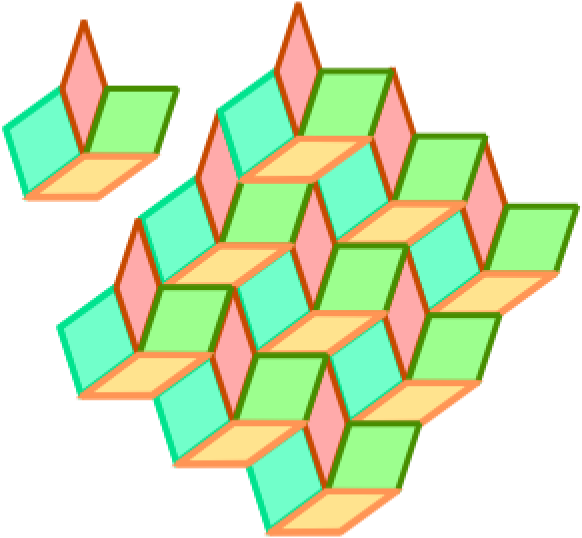
In 1974 Penrose proposed two rhombic prototiles whose angles are $72^\circ, 108^\circ$ or $36^\circ, 144^\circ$ and whose sides are decorated with red and green arrows. The tiles are to be assembled so that the color and direction of the arrows match for neighboring tiles.

Theorem (Penrose)

Every tiling of the plane by these rhombs satisfying the matching rule in non-periodic.

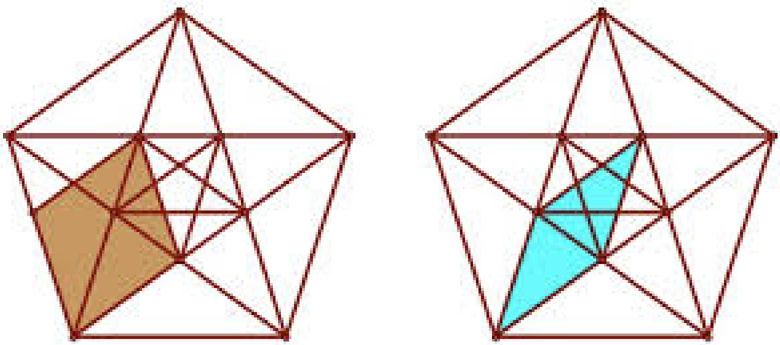
18. Penrose's rhombs with arrows.



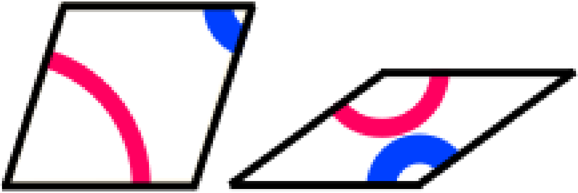


Of course, without the matching rule the rhombs may be assembled into a fundamental region which can be repeated to make a periodic tiling.

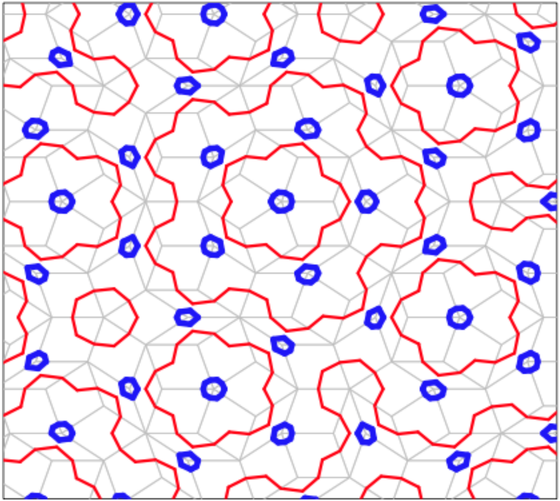
20. Penrose's rhombs are related to the pentagon



If you replace colored arrows by blue and red arcs, the matching rule is to extend the colored arcs continuously.

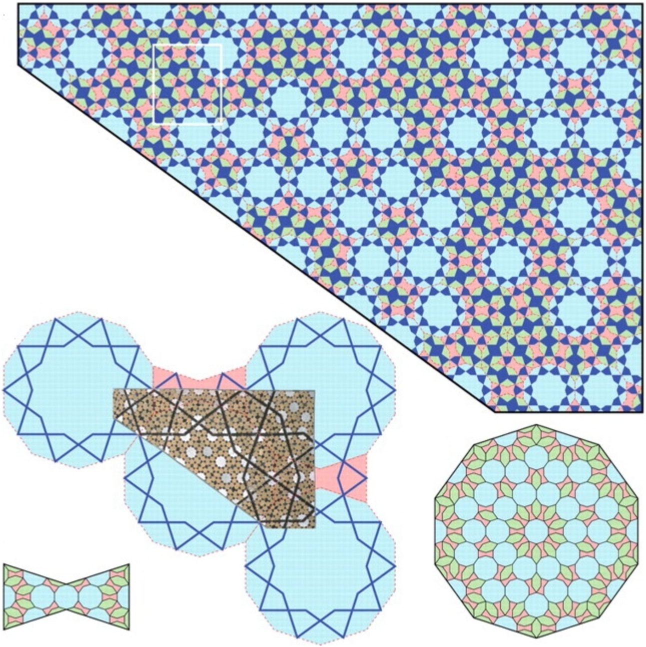


22. Another way of displaying the matching rule -

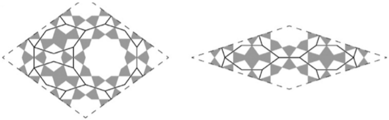


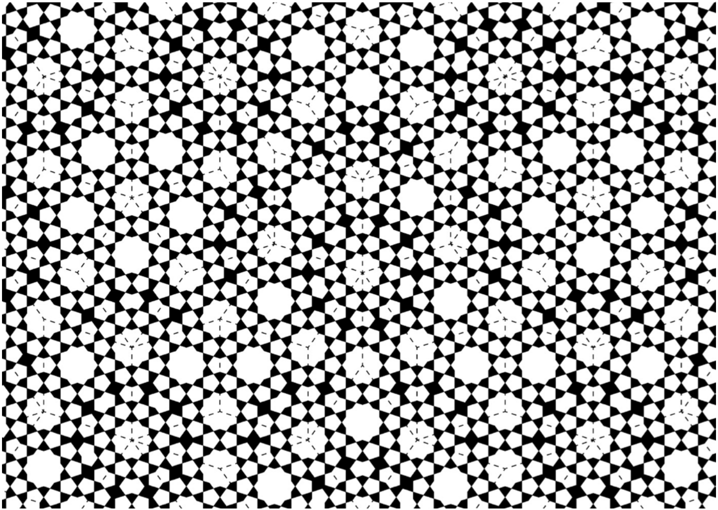
23. Quasiperiodic Tiling in Darb-i Imam shrine in Isfahan, Iran, 1453

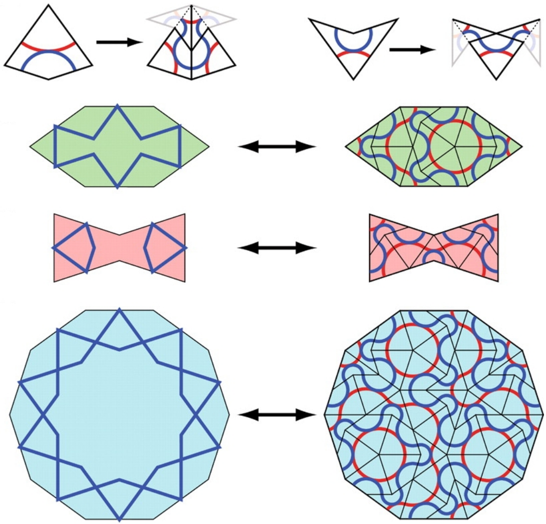
24. Bowties, Hexagons and Decagons are Girih prototiles

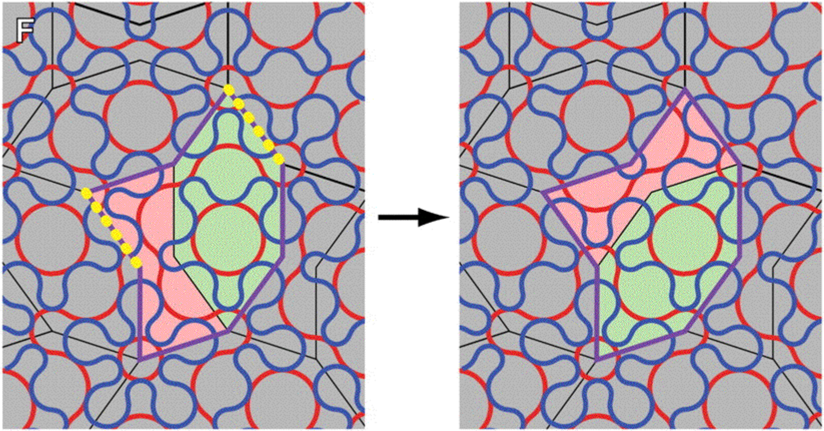


25. Cromwell's decorated rhombs to mimic Islamic tiling





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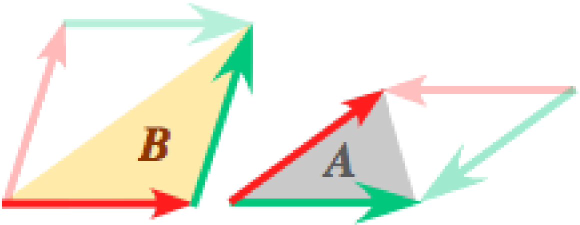
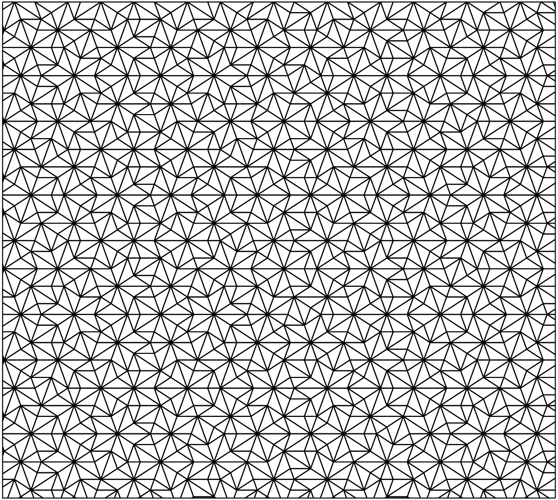
Penrose tiles can be subdivided into smaller Penrose tiles such that the resulting pattern is a Penrose tiling. This process is calles deflation. It may be used to produce a simple computer program to draw the tiling.

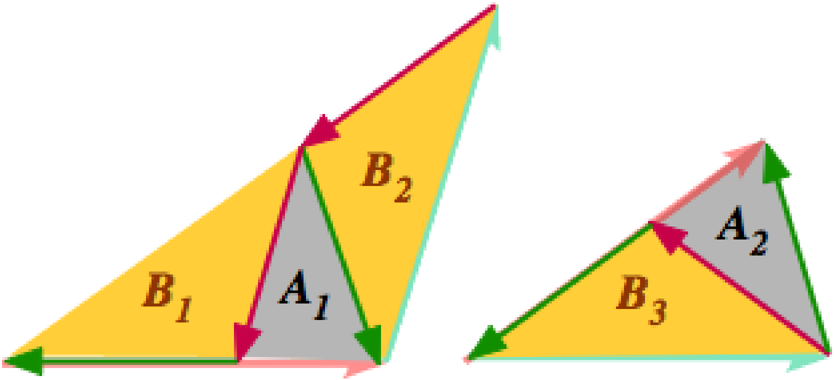


Consider half-rhombs. The result is a $36^\circ - 72^\circ - 72^\circ$ acute triangle called "A" and a $36^\circ - 108^\circ - 36^\circ$ obtuse triangle called "B." Drawing the colors and arrows of the half rhombs recaptures the rhombic tiling.

29. Tiling by halfrhombs



Each of these half-rhombs may be subdivided into two or three smaller half rhombs.

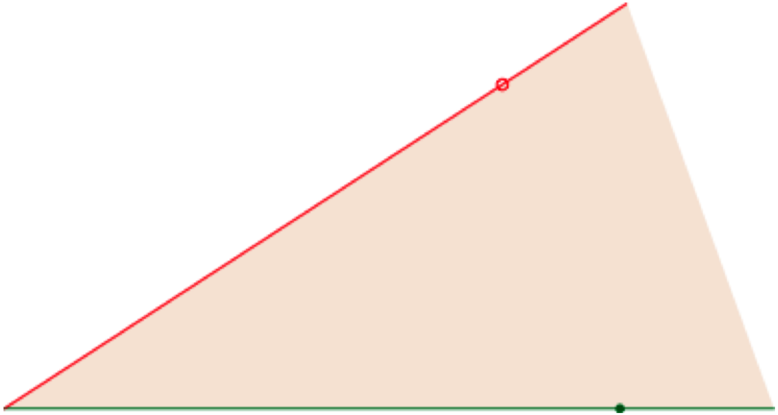


Side lengths of the new half rhombs are $\frac{1}{\phi} = \frac{\sqrt{5}-1}{2} \approx 0.618034$ of the original. $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618034$ is called the golden ratio.

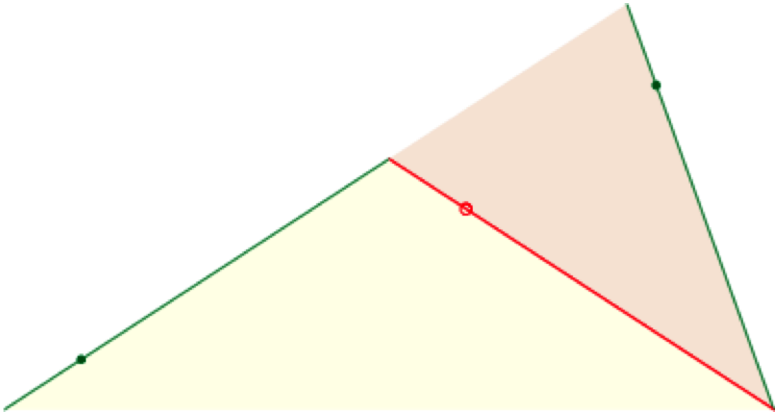
One starts with a half-rhomb, say A. Then one calls the subdivision recursively to a certain depth and then prints the colored halfrhombs and edges. Here is an entire elementary program written in **R**:

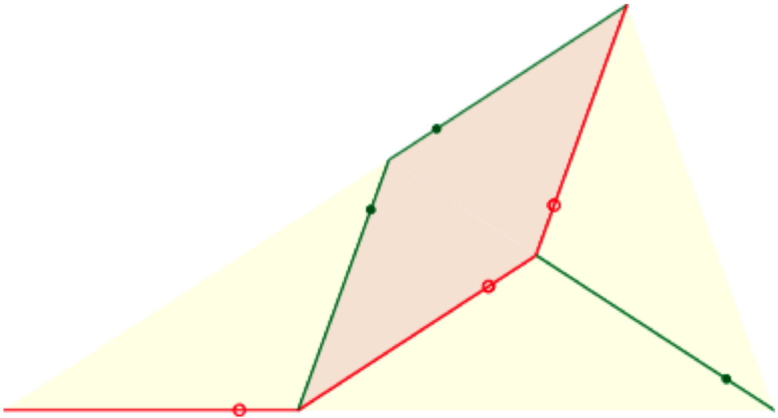
```
fh=.5*(sqrt(5)-1); fi=1-fh; rr=10*cos(pi/5); tt=10*sin(pi/5)

bug <- function(n,x,y,p,q,r,s) if(n>1) {lines(c(x+p,x+fh*(p+r)),c(y+q,y
+fh*(q+s))); lines(c(x+fh*p,x+fh*(p+r)),c(y+fh*q, y+fh*(q+s))); bug(n-1,
x+fh*(p+r),y+fh*(q+s),-fh*r,-fh*s,-fh*p,-fh*q); bug(n-1,x+p+r,y+q+s,-fi*
(p+r),-fi*(q+s),fi*p-fh*r,fi*q-fh*s); lil(n-1,x+fh*(p+r),y+fh*(q+s),
fi*p-fh*r,fi*q-fh*s,-fh*r,-fh*s)} else {polygon(c(x,x+p,x+p+r),c(y,y+q,y
+q+s),col=7,border=NA); lines(c(x,x+p,x+p+r),c(y,y+q,y+q+s))}

lil <- function(n,x,y,p,q,r,s) if(n>1) {lines(c(x+p,x+fh*r),c(y+q,y
+fh*s)); lil(n-1,x+p,y+q,r-p,s-q,fh*r-p,fh*s-q); bug(n-1,x+p,y+q,fh*r-p,
fh*s-q,-fh*r,-fh*s)} else {polygon(c(x,x+p,x+r),c(y,y+q,y
+s),col=2,border=NA); lines(c(x+r,x,x+p),c(y+s,y,y+q))}

plot(1, 1, xlim=c(5.2,8.6), ylim=c(0.18,3.58), type="n", xaxt="n",
yaxt="n", xlab="", ylab="")

lil(10,0,0,10,0,rr,tt)
```

32. Subdivision Depth 1

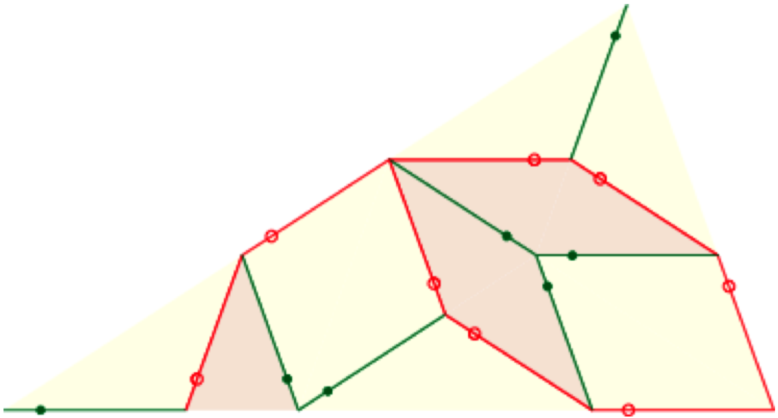


33. Subdivision Depth 2

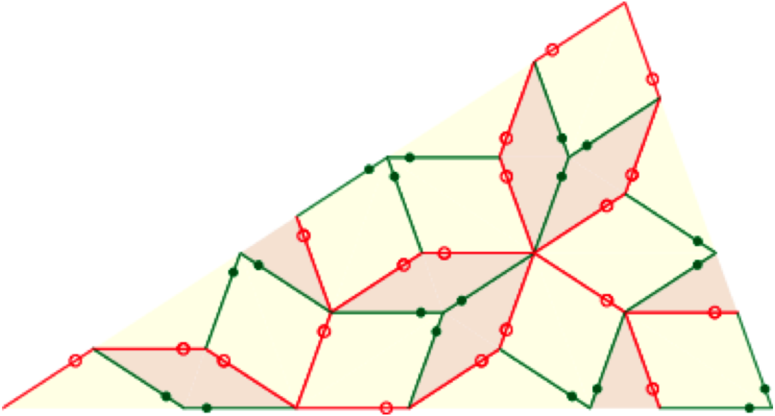




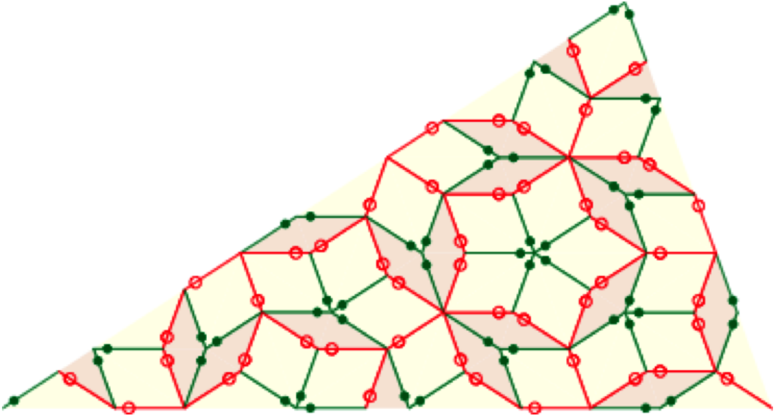
35. Subdivision Depth 4

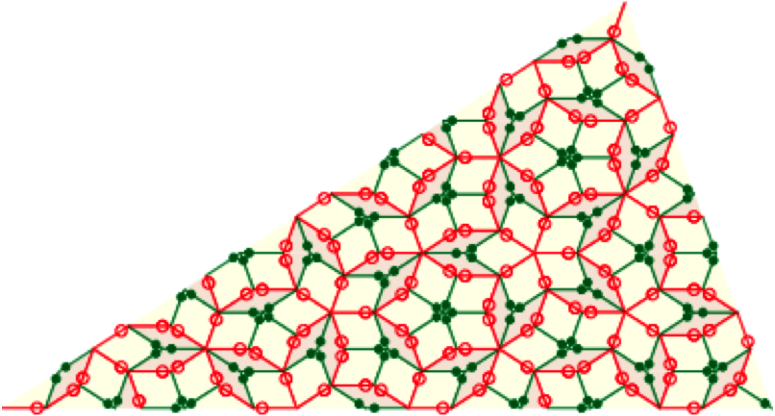


36. Subdivision Depth 5

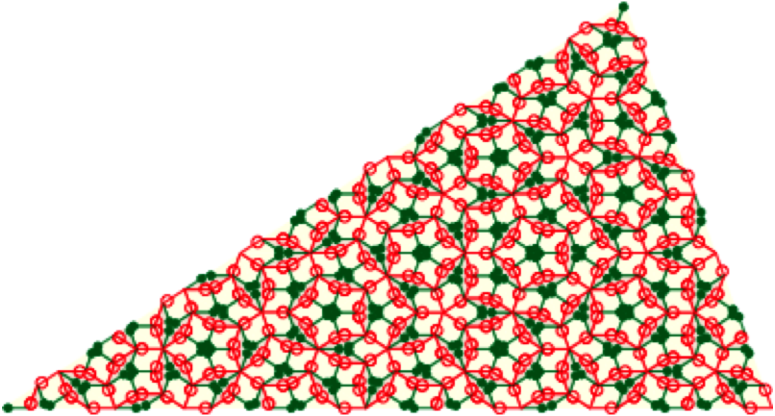


37. Subdivision Depth 6

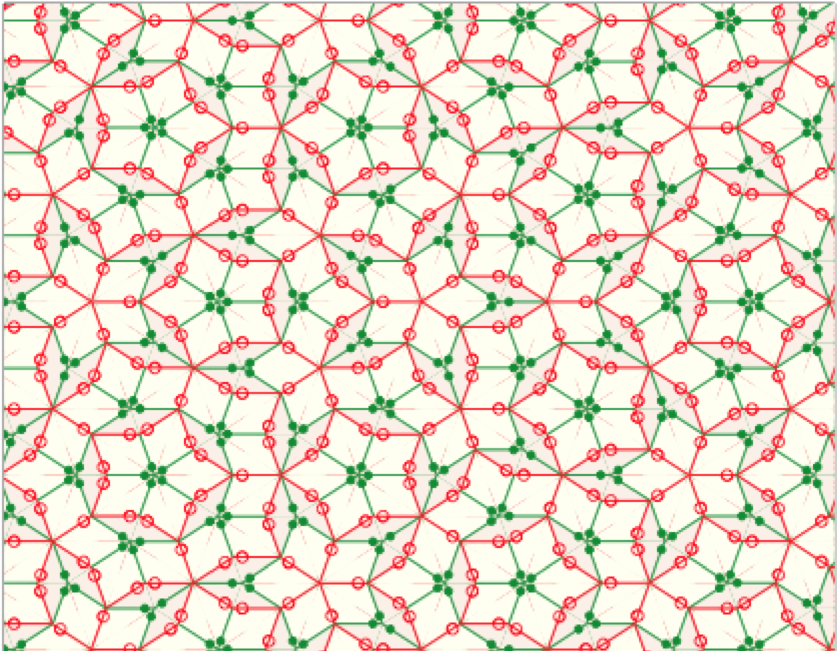




39. Subdivision Depth 8



40. Clip the pattern to a rectangle inside the triangle



Subdividing the original tile with unit sides makes a finer and finer tiling. The lengths of the sides have decreased by a factor of $\phi = \dfrac{1+\sqrt{5}}{2}$, the golden ratio each step. By scaling the n-th subdivision up by a factor of ϕ^n makes unit sided tiles filling in larger and larger triangles which cover larger and larger disks. The existence of the tiling of the plane follows from the theorem.

Theorem (Extension Theorem)

Let $\mathcal{S}$ be any finite set of prototiles, each of which is a closed topological disk. If $\mathcal{S}$ tiles over arbitrarily large circular disks D, then $\mathcal{S}$ admits a tiling over the plane.

The proof of this theorem depends on some notions from Math 3220. One selects a subsequence of tilings covering larger and larger disks that converges on compact subsets to a tiling of the plane.

While we're at it, let us argue that this Penrose tiling is non-periodic. Let us prove that the ratio of obtuse half-rhombs to acute half-rhombs is irrational. Then the tiling could not have been periodic because if it were, then there would be finitely many obtuse and acute half-rhombs in a fundamental region and the ratio would be rational. The asymptotic ratio on larger and larger triangles would be the same as this ratio on the fundamental region.

Theorem

In the Penrose tiling constructed above, the asymptotic ratio of obtuse to acute half-rhombs is ϕ, the golden ratio.

We give a proof using standard arguments from Math 3210.

Thus there are ϕ times more thick rhombs than thin rhombs in the penrose tiling!

43. Proof of the ratio.

Proof. Let a_n be the number of acute half-rhombs and b_n the number of obtuse half-rhombs in the nth stage of the subdivided triangle. We have $a_1 = 1$ and $b_1 = 0$. Subdividing an acute half-rhomb yields one acute and one obtuse half-rhomb at the next stage. Similarly, subdividing an obtuse half-rhomb yields two obtuse and one acute half-rhomb. Thus we get the recursion

$$\begin{aligned} a_{n+1} &= a_n + b_n \\ b_{n+1} &= a_n + 2b_n. \end{aligned}$$

Hence the ratio

$$r_{n+1} = \frac{b_{n+1}}{a_{n+1}} = \frac{a_n + 2b_n}{a_n + b_n} = \frac{1 + 2\frac{b_n}{a_n}}{1 + \frac{b_n}{a_n}} = \frac{1 + 2r_n}{1 + r_n}$$

so $r_1 = \frac{b_1}{a_1} = 0$ implies $r_2 = 1$. All other r_n's are positive also.

Note that the sequence is increasing. We show increments are positive. Arguing by induction, for the base case, $r_2 = 1 > r_1 = 0$. Assuming the induction hypothesis that the increment is positive for $n > 1$, the next increment is also positive

$$
\begin{aligned}
r_{n+2} - r_{n+1} &= \frac{1 + 2r_{n+1}}{1 + r_{n+1}} - \frac{1 + 2r_n}{1 + r_n} \\
&= \frac{(1 + r_n)(1 + 2r_{n+1}) - (1 + r_{n+1})(1 + 2r_n)}{(1 + r_{n+1})(1 + r_n)} \\
&= \frac{(1 + r_n + 2r_{n+1} + 2r_n r_{n+1}) - (1 + r_{n+1} + 2r_n + 2r_{n+1} r_n)}{(1 + r_{n+1})(1 + r_n)} \\
&= \frac{r_{n+1} - r_n}{(1 + r_{n+1})(1 + r_n)} > 0.
\end{aligned}
$$

The sequence is also bounded above:

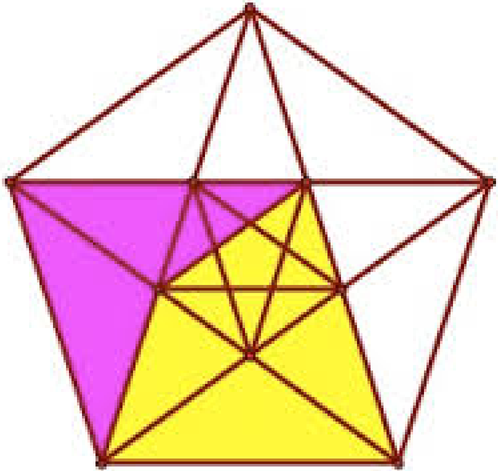
$$r_{n+1} = \frac{1+2r_n}{1+r_n} \leq \frac{2+2r_n}{1+r_n} = 2.$$

But a bounded monotone increasing sequence is convergnt. Thus $r = \lim_{n\to\infty} r_n$ exists.

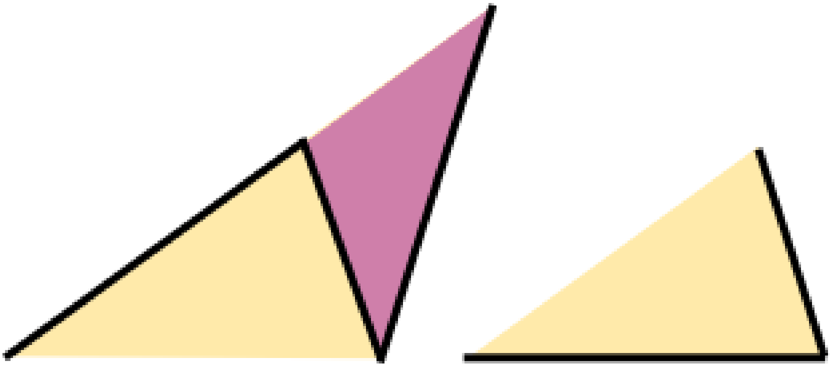
Taking the limit of both sides of the recursion as $n \to \infty$,

$$r = \frac{1+2r}{1+r}$$

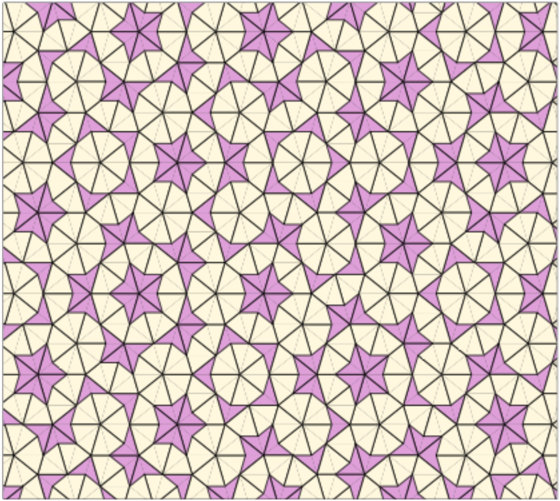
which implies $r = \phi$. $\square$



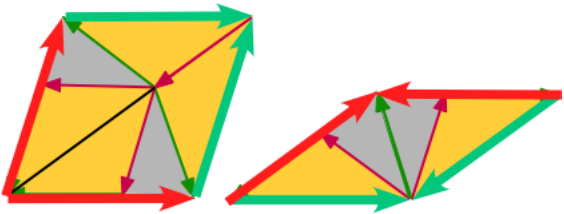
Penrose's kites and darts tiling and the rhombic tiling are closely related. In fact, if the half-rhombs are colored like this



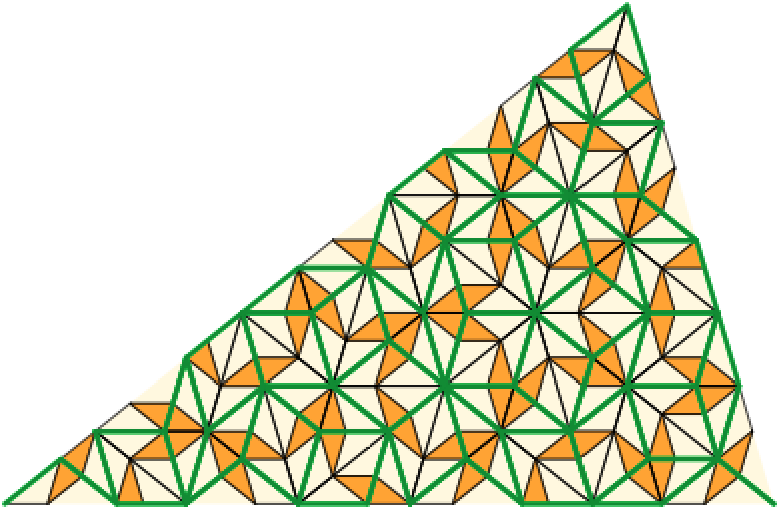
48. Kites and Darts -



The subdivision may be done in reverse for any Penrose tiling of the plane by rhombs. The half-rhombs in any tiling fit uniquely into one of the following patches.



The result is a coarser tiling in pure red and green for which the matching conditions also hold. The new side lengths are stretched by a factor ϕ. The coarser tiling by rhombs is called the inflated tiling.

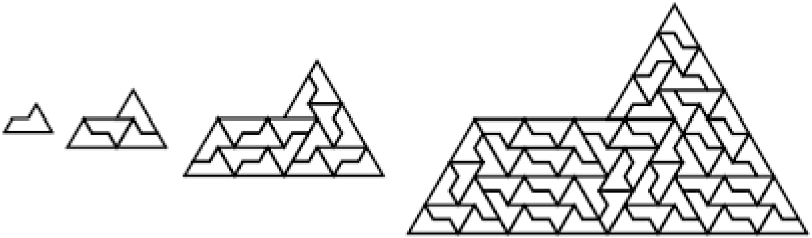


Theorem (Penrose)

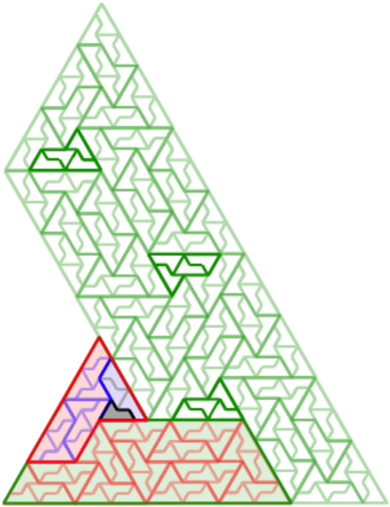
Any tiling of the plane by Penrose rhombs is non-periodic.

Proof. If any tiling of the plane by Penrose rhombs were periodic, then the entire tiling would coincide with the translation of itself by a finite length. Then the inflated tiling by rhombs, being uniquely determined, would have to be periodic with the same period as well. But the rhombs in the inflated tiling are larger by a linear factor ϕ. By inflating each inflation, one obtains a hierarchy of tilings by exponentially larger and larger rhombs. The size of the tiles is eventually much larger than the length of the translation. It is therefore impossible that the original translation be a symmetry of that inflated tiling. $\square$

52. Sphinx tile

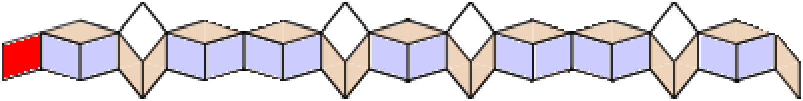


53. Sphinx nonperiodic tiling

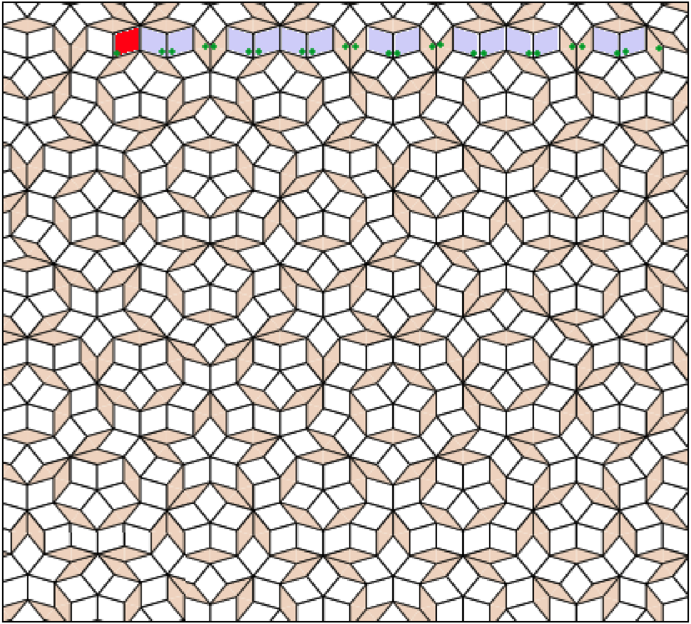


Because a Penrose tiling of the plane has the hierarchical inflation structure, one cannot just start from a seed and attach Penrose tiles arbitrarily and hope that the tiling will grow without bound. Tiles have to be consistent with the inflation structure.

Consider the following strip taken from a tiling of the plane.

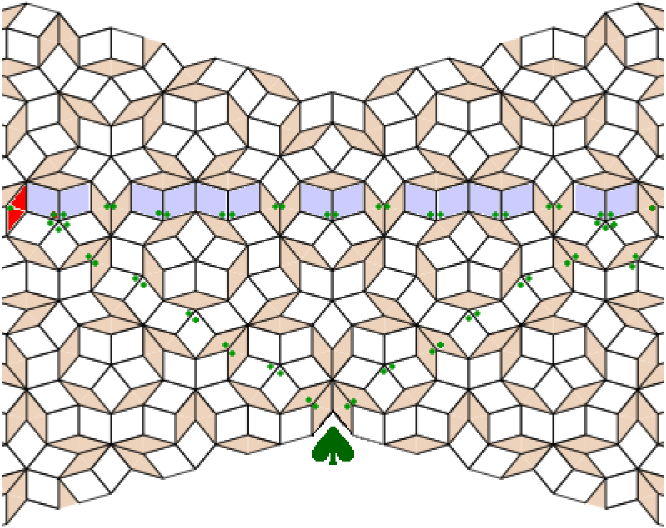


55. Strip is part of tiling of plane.



56. Change one tile and the tiling does not continue to the whole plane.

Change one tile at the end of the strip. The tiling cannot be continued beyond the spade. We indicate the green arrow vertex of the rhomb with a green dot.



It is impossible to continue tiling at the 72° angle at the green spade. It must be filled by one thick or two thin rhombs. But the arrows can't be matched.

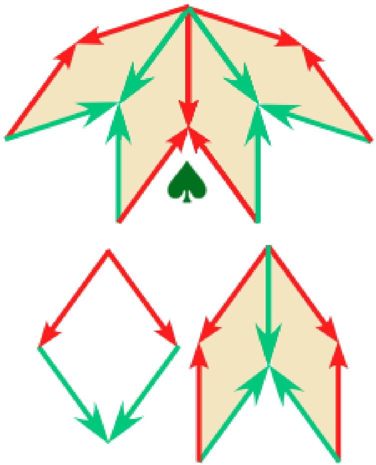
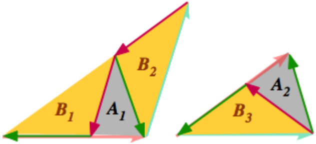


Figure: de Bruijn 1918-2012

After graduating from Leiden U. in 1941, de Bruijn was made assistant to the Math. Dept. at TU Delft so avoided Nazi forced labor.

In 1943 he completed his PhD at the Vrije U in Amsterdam writing about modular forms. He also worked for Philips Research company doing optimal control. In 1952 he moved to Amsterdam U where he was asked to teach a course in applied mathematics and developed a famous text on asymptotic methods. In 1960 he moved to TU Eindhoven.

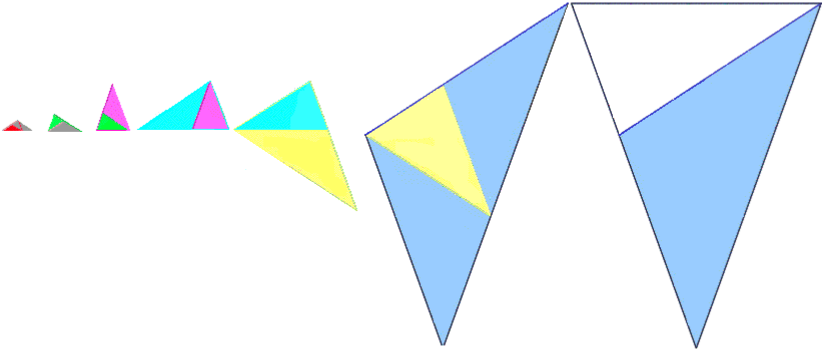
Trained as a pure mathematician, de Bruijn worked also in several areas of applied math ranging from optimization, combinatorics and automated proof.

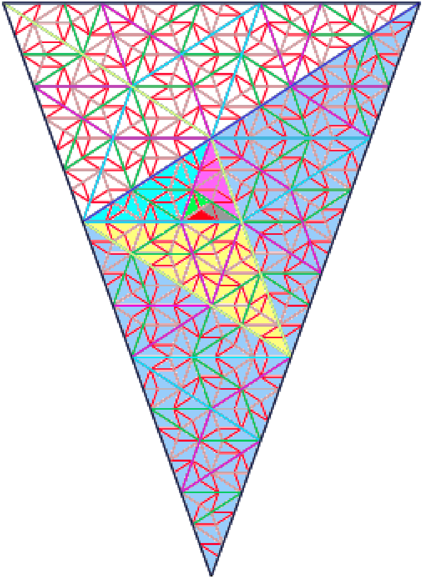


Start with a red obtuse halfrhomb, choose a halfrhomb that contains it.

It fits in, say, an obtuse gray halfrhomb in the B_1 position. The gray halfrhomb fits in the acute lime halhrhomb in the B_3 position. The lime halfrhomb fits in the acute magenta halfrhomb in the A_2 position. We get a sequence

$$B_1, B_3, A_2, A_2, A_2, A_1, B_3, \ldots$$

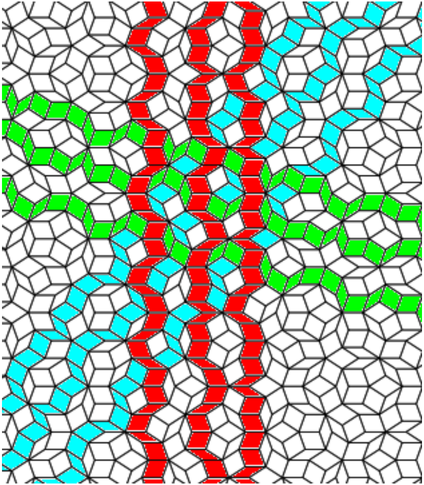




Use deflation to subdivide the n-th halfrhomb n times. The result is a tiling T_n of a neighborhood of the original red halfrhomb.

The sequence can be chosen arbitrarily, except that A_1, B_1 or B_2 must be followed by B_1, B_2 or B_3 and A_2 or B_3 must be followed by A_1 or A_2. The tilings are "eventually constant" so converge to a tiling of the plane.

For most "good" sequences, T_n fills up the plane. The red tile may remain at the edge of all T_n for exceptional sequences. There are many uncountably many "good" sequences. It turns out there are uncountably many tilings.



Opposite sides of a rhomb are parallel. We may form a ribbon by attaching rhombs along opposite sides. There are five directions taken by sides, so there are five families of ribbons that do not intersect, determined by the side directions.

de Bruijn discovered that the pattern may be reconstructed from families of parallel lines, called the pentagrid.

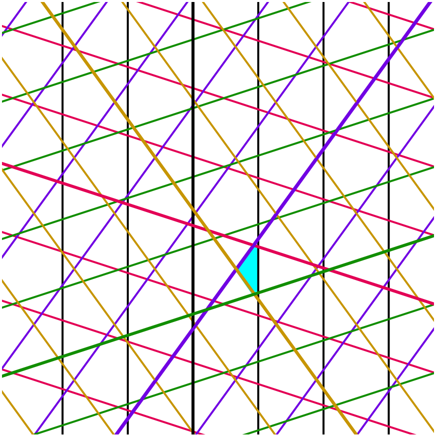


Figure: de Bruijn's Pentagrid

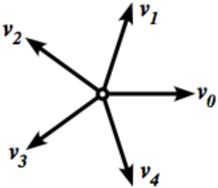
Let $\gamma_0, \ldots, \gamma_4$ be real numbers satisfying $\gamma_0 + \cdots + \gamma_4 = 0$. We consider five grids in the plane

$$\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{v}_j + \gamma_j \in \mathbb{Z}\},$$

where for $j = 0, \ldots, 4$,

$$\mathbf{v}_j = \big(\cos(2\pi j/5), \sin(2\pi j/5)\big)$$

is a unit vector.



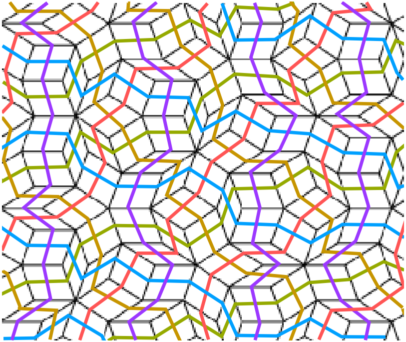


Figure: Skeleton curves through midpoints of edges

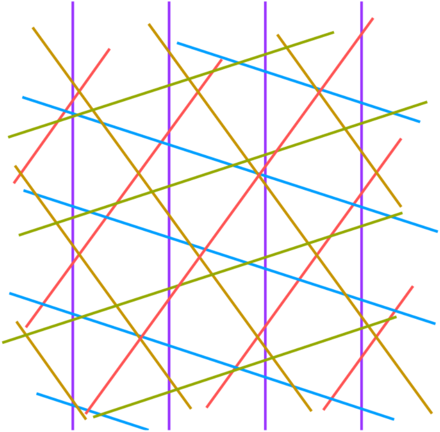


Figure: Corresponding Pentagrid

The grids are regular if no point of $\mathbf{R}^2$ belongs to more than two of the five grids. Almost all choices of $\gamma_0, \ldots, \gamma_4$ make a regular pentagrid.

Given such $\gamma_0, \ldots, \gamma_4$ real numbers, we associate five integers $K_0(\mathbf{x}), \ldots, K_4(\mathbf{x})$ where, using the ceiling function $\lceil \bullet \rceil$ (least integer n such that $\bullet \leq n$),

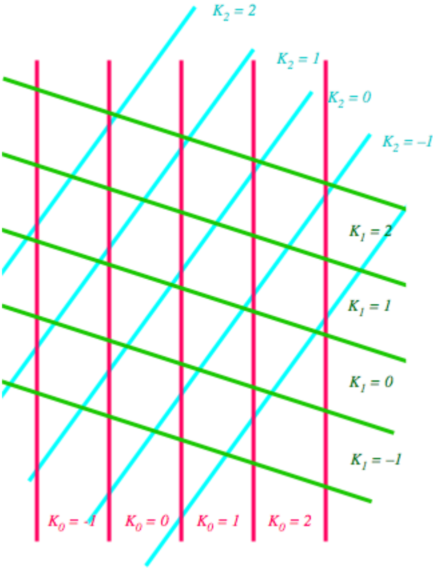
$$K_j(\mathbf{x}) = \lceil \mathbf{v}_j \cdot \mathbf{x} + \gamma_j \rceil. \tag{1}$$

$K_0(\mathbf{x}), \ldots, K_4(\mathbf{x})$ are "pentagrid coordinates of a rhombus in the tiling." The vertices of the rhombuses are given by

$$f(\mathbf{x}) = \sum_{j=0}^{4} K_j(\mathbf{x})\mathbf{v}_j, \qquad \text{as } \mathbf{x} \text{ runs through all points of } \mathbf{R}^2.$$

$f(\mathbf{x})$ is constant on regions between the grid lines.

65. Pentagrid coordinates



Let r and s be integers such that $0 \leq r < s \leq 4$ and $k_r, k_s \in \mathbb{Z}$. The point $\mathbf{x}_0$ determined by the equations

$$\mathbf{v}_r \cdot \mathbf{x} + \gamma_r = k_r, \qquad \mathbf{v}_s \cdot \mathbf{x} + \gamma_s = k_s$$

is the intersection point of a line in the r-th grid and the s-th grid. In a small neighborhood of $\mathbf{x}_0$, $f(\mathbf{x})$ takes the values of four vertices of a rhomb. *e.g.*, if $r = 0$, $s = 1$ the four vertices correspond to

$$\big(K_0(\mathbf{x}_0), \ldots, K_4(\mathbf{x}_0)\big) + \big(\epsilon_1, \epsilon_1, 0, 0, 0\big)$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$.

Thus the intersection points of the pentagrid lines correspond to rhombs. The regions between the grid lines correspond to vertices, whose positions are given by $f(\mathbf{x})$.

Theorem (de Bruijn 1981)

The rhombuses corresponding to a regular or singular pentagrid form a tiling of the plane by Penrose's rhombic prototiles. An orientation and color for the sides can be determined from the pentagrid such that the Penrose matching conditions hold.

Every Penrose tiling of the plane comes from a regular or singular pentagrid.

Not all pentagrid coordinates $k_0, \ldots, k_4$ occur in a tiling. de Bruijn gives a geometric interpretation of the ones that do.

68. Index of a vertex and how to color the rhombs.

In a regular pentagrid, for $\mathbf{x} \in \mathbf{R}^2$ we rewrite (1)

$$K_j(\mathbf{x}) = \mathbf{v}_j \cdot \mathbf{x} + \gamma_j + \lambda_j(z). \tag{2}$$

where $0 \leq \lambda_j(z) < 1$. Since the pentagrid is regular, at most two of the $\lambda_0(z), \ldots, \lambda_4(x)$ are equal to zero, hence

$$0 < \lambda_0(z) + \cdots + \lambda_4(x) < 5.$$

On the other hand, since we assumend $\sum_j \gamma_j = 0$, thus

$$\sum_{j=0}^4 K_j(z) = \left(\sum_{j=0}^4 \mathbf{v}_j\right) \cdot \mathbf{x} + \sum_{j=0}^4 \gamma_j + \sum_{j=0}^4 \lambda_j(z) = \sum_{j=0}^4 \lambda_j(z)$$

The left side is an integer between zero and five. This quantity defines the index of a vertex

$$\mathrm{Ind}(z) = \sum_{j=0}^4 K_j(z) \in \{1, 2, 3, 4\}$$

Every vertex in the rhombic tiling may be represented as

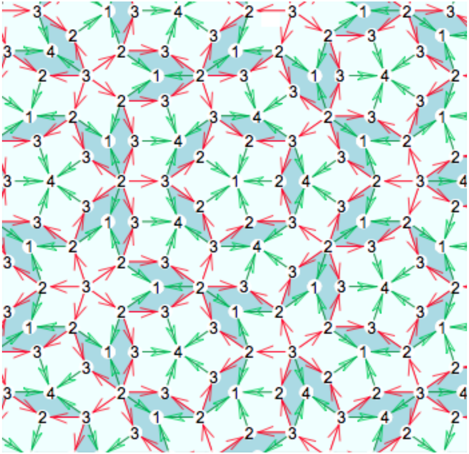
$$k_0\mathbf{v}_0 + \cdots + k_4\mathbf{v}_4$$

where integers k_j satisfy $\sum_{j=0}^4 k_j \in \{1, 2, 3, 4\}$.

In fact, an index may be assigned to any Penrose tiling. If we move a point along the edge of a rhobus, the index increases by one in the directions $\mathbf{v}_0, \ldots, \mathbf{v}_4$ and deacreases by one in the directions $-\mathbf{v}_0, \ldots, -\mathbf{v}_4$.

Hence, a thick rhombus either has index values 1 and 3 at the 72° angles and 2 at the 108° angles, or 2 and 4 at the 72° angles and 3 at the 108° angles. A thin rhombus either has index values 1 and 3 at the 144° angles and 2 at the 36°, or 2 and 4 at the 144° and 3 at the 36°. Edges connecting a point of index 3 to a point of index 2 are colored red, edges connecting a 1 to a 2 or a 3 to a 4 are green. Green arrows point from 2 to 1 or from 3 to 4. Orienting red edges is similar but a bit more involved.

70. Index of a vertex and how to color the rhombs. - -



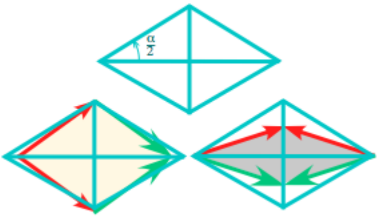


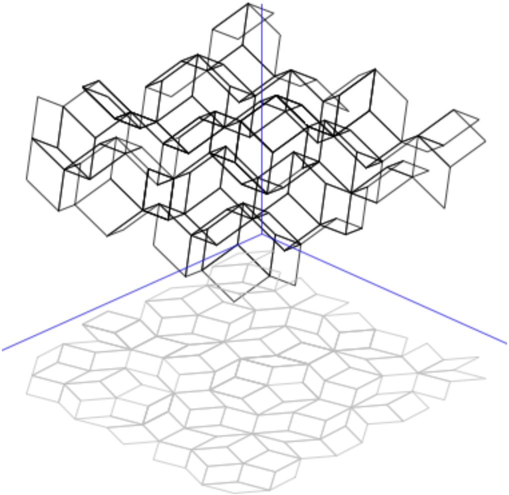
Figure: Wieringa rhomb

R. M. A. Wieringa noticed that the thick and thin rhombs are in fact projections of a single rhomb in 3d. Take a rhomb whose short diagonal equals the short diagonal of the thick rhomb and whose long diagonal is the long diagonal of the thin rhomb.

Using the x and y coordinates of the tiling and half of the index as the z-coordinate gives a faceted surface of thickness 3 called a Wieringa roof.

Note that the endpoints of the short diagonal of the thick rhomb and the endpoints of the long diagonal of the thin rhomb always have the same index.

72. Wire frame Wieringa roof of previous tiling



The cosines can be computed. If s and c denote sine and cosine of $\theta = 72^\circ$, then

$$1 = e^{5\theta i} = (c+is)^5 = c^5 + 5c^4si - 10c^3s^2 - 10c^2s^3i + 5cs^4 + s^5i.$$

Using $s^2 = 1 - c^2$, the imaginary part is

$$\begin{aligned} 0 &= 5c^4s - 10c^2s^3 + s^5 \\ &= s\left[5c^4 - 10c^2(1-c^2) + (1 - 2c^2 + c^4)\right] \\ &= s\left[16c^4 - 12c^2 + 1\right] \end{aligned}$$

so

$$c^2 = \tfrac{12 \pm \sqrt{144-64}}{32} = \tfrac{3 \pm \sqrt{5}}{8}.$$

But $\cos 72^\circ < \cos 60^\circ = .5$ so we take the "$-$". In fact

$$c^2 = \tfrac{3-\sqrt{5}}{8} = \left(\tfrac{\sqrt{5}-1}{4}\right)^2.$$

Using the double angle formula

$$\cos 72^\circ = \cos^2 36^\circ - \sin^2 36^\circ = 2\cos^2 36^\circ - 1$$

Thus

$$\cos^2 36^\circ = \tfrac{1}{2}\left(1 + \cos 72^\circ\right) = \tfrac{1}{2}\left(1 + \tfrac{\sqrt{5}-1}{4}\right) = \tfrac{3+\sqrt{5}}{8} = \left(\tfrac{1+\sqrt{5}}{4}\right)^2.$$

Let's compute the length of the long diagonals of D_1 of the thick rhomb and D_2 of the thin rhomb, assuming the side has unit length.

$$D_1^2 = (1 + \cos 72^\circ)^2 + \sin^2 72^\circ = 2 + 2\cos 72^\circ = 2 + \tfrac{\sqrt{5}-1}{2} = \tfrac{3+\sqrt{5}}{2}$$
$$D_2^2 = (1 + \cos 36^\circ)^2 + \sin^2 36^\circ = 2 + 2\cos 36^\circ = 2 + \tfrac{1+\sqrt{5}}{2} = \tfrac{5+\sqrt{5}}{2}$$

Corner to corner on the thick rhomb has a run of D_1 and a rise of 1 so that

$$1 + D_1^2 = 1 + \tfrac{3+\sqrt{5}}{2} = \tfrac{5+\sqrt{5}}{2} = D_2^2.$$

Thus the Pythagorean Theorem holds: the hypotenuse D_2, the diagonal of the Weiringa rhomb projects to horizontal, the D_1 diagonal of the thick rhomb. The other direction is similar.

Let's compute the length of the short diagonals of d_1 of the thick rhomb and d_2 of the thin rhomb.

$$d_1^2 = (1 - \cos 72^\circ)^2 + \sin^2 72^\circ = 2 - 2\cos 72^\circ = 2 - \tfrac{\sqrt{5}-1}{2} = \tfrac{5-\sqrt{5}}{2}$$
$$d_2^2 = (1 - \cos 36^\circ)^2 + \sin^2 36^\circ = 2 - 2\cos 36^\circ = 2 - \tfrac{1+\sqrt{5}}{2} = \tfrac{3-\sqrt{5}}{2}$$

Corner to corner on the thin rhomb has a run of d_2 and a rise of 1 so that

$$1 + d_2^2 = 1 + \tfrac{3-\sqrt{5}}{2} = \tfrac{5-\sqrt{5}}{2} = d_1^2.$$

Thus the Pythagorean Theorem holds: the hypotenuse d_1, the diagonal of the Weiringa rhomb projects to horizontal, the d_2 diagonal of the thick rhomb.

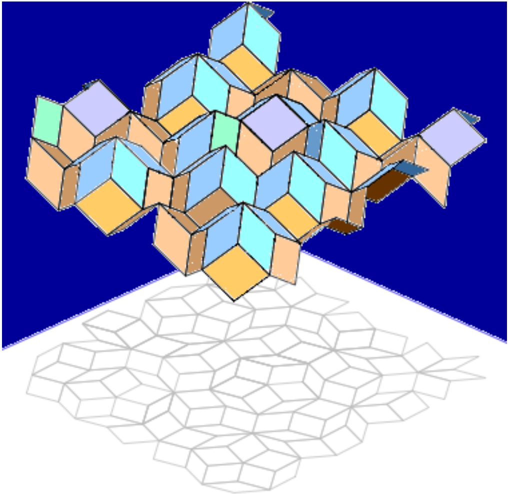
The Wieringa rhombus acute angle satisfies

$$\tan^2\frac{\alpha}{2}=\frac{d_1^2}{D_2^2}=\frac{5-\sqrt{5}}{5+\sqrt{5}},\qquad 1-\tan^2\frac{\alpha}{2}=\frac{2\sqrt{5}}{5+\sqrt{5}}$$

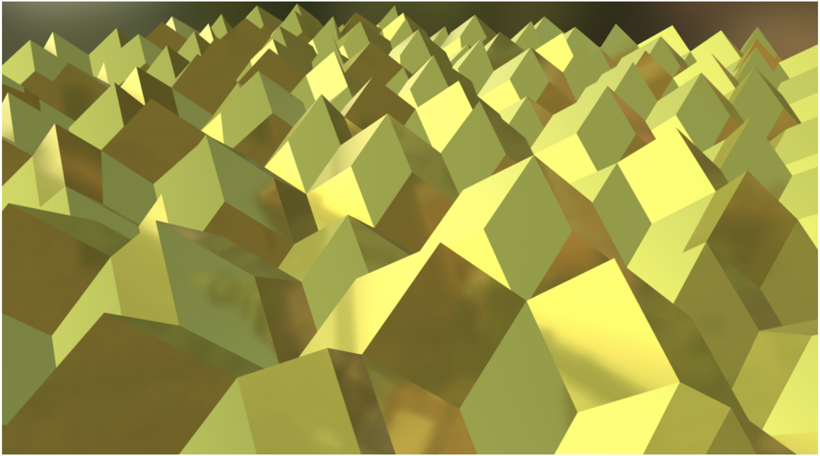
so that by the double angle formula,

$$\tan^2\alpha=\frac{4\tan^2\frac{\alpha}{2}}{\left(1-\tan^2\frac{\alpha}{2}\right)^2}=\frac{\frac{4(5-\sqrt{5})}{5+\sqrt{5}}}{\frac{20}{(5+\sqrt{5})^2}}=\frac{(5-\sqrt{5})(5+\sqrt{5})}{5}=4.$$

Thus $\tan\alpha=2$ and $\alpha=63.43495^\circ$.



78. Wieringa roof found on the internet



79. Projection of lattice points from $\mathbf{R}^5$

de Bruijn 1981 noticed that the vertices of the Penrose tiling is a projection of lattice points in $\mathbf{R}^5$ to a certain two plane. Divide five dimensional space into unit cubes in the standard way (vertices are points with integral coordinates). Corresponding to five integers $k_0, \ldots, k_4$ is an open cube given by the set of all points $(z_0, \ldots, z_4)$ with

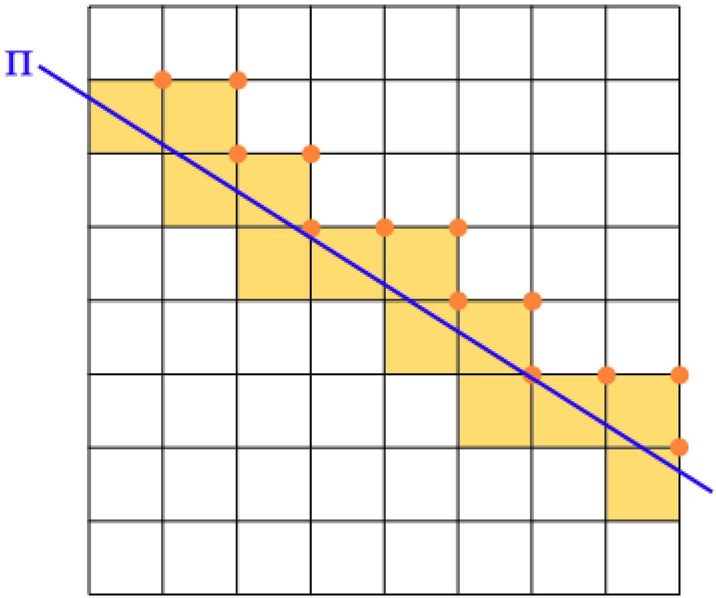
$$k_0 < z_0 < k_0, \qquad \ldots, \qquad k_4 < z_4 < k_4.$$

Consider the two dimensional plane Π given by the three real equations

$$\sum_{j=0}^{4} z_j = 0, \qquad \sum_{j=0}^{4} (z_j - \gamma_j)\mathbf{v}_{2j} = 0. \tag{3}$$

Theorem (de Bruijn)

The vertices of tiling produced by a regular pentagrid with parameters $\gamma_0, \ldots, \gamma_4$ are the points $\sum_{j=0}^{4} k_j \mathbf{v}_j$ where $(k_0, \ldots, k_4)$ runs through all elements of $\mathbb{Z}^5$ whose open cube has nonempty intersection with the plane Π given by (3).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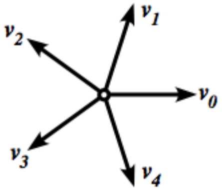

Writing $c_j = \cos(2\pi j/5)$ and $s_j = \sin(2\pi j/5)$, the five directions vectors are $\mathbf{v}_0 = (1, 0)$, $\mathbf{v}_1 = (c_1, s_1)$, $\mathbf{v}_2 = (c_2, s_2)$, $\mathbf{v}_3 = (c_3, s_3) = (c_2, -s_2)$ and $\mathbf{v}_4 = (c_4, s_4) = (c_1, -s_1)$.

For a point (x, y) in the pentagrid, equations (2) become

$$k_j = c_j x + s_j y + \gamma_j + \lambda_j$$

where $0 \le \lambda_j < 1$ so the cube at $(k_0, \ldots, k_4)$ meets the parameterized plane

$$\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = x \begin{pmatrix} 1 \\ c_1 \\ c_2 \\ c_2 \\ c_1 \end{pmatrix} + y \begin{pmatrix} 0 \\ s_1 \\ s_2 \\ -s_2 \\ -s_1 \end{pmatrix} + \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}$$

A little more careful argument says it meets the open cube.

In other words, the vectors

$$\begin{pmatrix} z_0 - \gamma_0 \\ z_1 - \gamma_1 \\ z_2 - \gamma_2 \\ z_3 - \gamma_3 \\ z_4 - \gamma_4 \end{pmatrix} \quad \text{are in the subspace spanned by} \quad \left\{ \begin{pmatrix} 1 \\ c_1 \\ c_2 \\ c_2 \\ c_1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \\ s_2 \\ -s_2 \\ -s_1 \end{pmatrix} \right\}$$

and thus are perpendicular to the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ c_2 \\ c_1 \\ c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ -s_1 \\ s_1 \\ -s_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ c_2 \\ c_4 \\ c_6 \\ c_8 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ s_4 \\ s_6 \\ s_8 \end{pmatrix} \right\}$$

We check the only non-obvious inner product between the first and second (cosine) vectors. Recall that $c_1 = \frac{\sqrt{5}-1}{4}$ so by the double angle formula $c_2 = 2c_1^2 - 1 = 1\left(\frac{\sqrt{5}-1}{4}\right)^2 - 1 = \frac{-1-\sqrt{5}}{4}$. Thus

$$1 + 4c_1c_2 = 1 + 4\left(\frac{-1+\sqrt{5}}{4}\right)\left(\frac{-1-\sqrt{5}}{4}\right) = 1 + 4\left(\frac{1-5}{16}\right) = 0.$$

Thus, because $\sum_{j=0}^4 \gamma_j = 0$ we have

$$0 = \sum_{j=0}^{4}(z_j - \gamma_j) = \sum_{j=0}^{4} z_j = 0$$

and

$$\mathbf{0} = \sum_{j=0}^{4}(z_j - \gamma_j)\mathbf{v}_{2j}$$

which proves tiling vertices coming from the pentagrid correspond to $(k_0, \ldots, k_4)$ for open cubes meeting the plane. The argument also works the other way around, proving the theorem. $\square$

Which pentagrids give distinct tilings? Translating the pentagrids by z_0 yields the same tiling. If $\gamma_0^*, \dots, \gamma_4^*$ are parameters for another pentagid then they produce a shifted pentagrid if $f(z) = f^*(z + z_0)$. *i.e.,* if there is a vector z_0 such that

$$z_0 \cdot \mathbf{v}_j + \gamma_j - \gamma_j^* \in \mathbb{Z}, \qquad \text{for all } j = 0, \dots, 4.$$

The vertices of the tiling depend only on the vector in (3),

$$\xi = \textstyle\sum_{j=0}^4 \gamma_j \mathbf{v}_{2j},$$

thus it determines whether the tilings are translates.

Theorem (de Bruijn)

Let $\gamma_0, \dots, \gamma_4$ and $\gamma_0^*, \dots, \gamma_4^*$ parameterize regular pentagrids. The tilings they determine are equal if and only if $\xi = \xi^*$.
The tilings are shift equivalent if and only if

$$\xi - \xi^* = \textstyle\sum_{j=0}^4 n_j \mathbf{v}_j, \quad \textit{for some } (n_0, \dots, n_4) \in \mathbb{Z}^5 \textit{ with } \sum_{j=0}^4 n_j = 0.$$

Almost all choices of $\gamma_0, \ldots, \gamma_4$ gave regular pentagrids, thus

$$\mathcal{R} = \left\{ \xi \in \mathbf{R}^2 : \ \xi \text{ correspond to a regular pentagrid} \right\}$$

consist of almost all points of $\mathbf{R}^2$.

Since there are only countably many points in the ideal $\mathcal{P} \in \mathbf{R}^2$, where

$$\mathcal{P} = \left\{ \sum_{j=0}^{4} n_j \mathbf{v}_j : \quad (n_0, \ldots, n_4) \in \mathbb{Z}^5 \text{ with } \sum_{j=0}^{4} n_j = 0. \right\},$$

then there are uncountably many equivalence classes of $\mathcal{R}/\mathcal{P}$, none of whose tiling is shift equivalent to to any other.

Thanks!