THE INTERMEDIATE VALUE THEOREM

Theorem. *Let f be continuous on $[a, b]$ then for any y such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$ there is a point $c \in (a, b)$ such that $f(c) = y$.*
In other words: every point between $f(a)$ and $f(b)$ is an image of a point in (a, b).

Proof. Without loss of generality let us assume $f(a) < f(b)$. Let y be such that $f(a) < y < f(b)$. We must prove that there exists a $c \in (a, b)$ such that $f(c) = y$.

Claim. *There is a sequence of nested sequence of intervals:*

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \ldots$$

such that:

- i $f(a_n) < y < f(b_n)$ *for all* $n \geq 0$
- ii $b_n - a_n = \frac{b-a}{2^n}$ *for all* $n \geq 0$

or we can find a point c such that $f(c) = y$

Proof of claim. We construct the sequence by induction.
The First Step: Set $a_0 = a$ and $b_0 = b$. Then [i] is satisfied by the assumption that $f(a) < y < f(b)$ and [ii] translates to $b_0 - a_0 = \frac{b-a}{2^0}$ which is also true by the definition of a_0, b_0.
Induction Hypothesis: Suppose we've constructed: $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ such that:

$$[a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n]$$

and:

- i $f(a_k) < y < f(b_k)$ for all $0 \leq k \leq n$
- ii $b_k - a_k = \frac{b-a}{2^k}$ for all $0 \leq k \leq n$

The $n + 1$-st step: We must construct the $n + 1$st interval so that the properties still hold. Let $d_n = \frac{a_n + b_n}{2}$ (the midpoint of the interval $[a_n, b_n]$). There are three cases and we will define the next interval accordingly:

$f(d_n) = y$: In this case we've found a source for y and we're done.

$f(d_n) > y$: Define $a_{n+1} = a_n$ and $b_{n+1} = d_n$. Since $b_{n+1} < b_n$ we get $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$. We must check:

- i $f(a_{n+1}) \overset{?}{<} y \overset{?}{<} f(b_{n+1})$
  $f(a_{n+1}) = f(a_n) < y$ by the induction hypothesis, and $y < f(d_n) = f(b_{n+1})$ because this is the case we're in.

ii $b_{n+1} - a_{n+1} \stackrel{?}{=} \frac{b-a}{2^{n+1}}$
$b_{n+1} - a_{n+1} = d_n - a_n = \frac{a_n+b_n}{2} - a_n = \frac{b_n-a_n}{2}$ by the induction hypothesis $b_n - a_n = \frac{b-a}{2^n}$ so $b_{n+1} - a_{n+1} = \frac{b-a}{2^{n+1}}$ as we needed to show.

$f(d_n) < y$: Define $a_{n+1} = d_n$ and $b_{n+1} = b_n$. Since $a_{n+1} > a_n$ then $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$. We must check:

i $f(a_{n+1}) \stackrel{?}{<} y \stackrel{?}{<} f(b_{n+1})$
$f(b_{n+1}) = f(b_n) > y$ by the induction hypothesis, and $y < f(d_n) = f(a_{n+1})$ because this is the case we're in.

ii $b_{n+1} - a_{n+1} \stackrel{?}{=} \frac{b-a}{2^{n+1}}$
$b_{n+1} - a_{n+1} = b_n - d_n = b_n - \frac{a_n+b_n}{2} = \frac{b_n-a_n}{2}$ by the induction hypothesis $b_n - a_n = \frac{b-a}{2^n}$ so $b_{n+1} - a_{n+1} = \frac{b-a}{2^{n+1}}$ as we needed to show.

$\square$

This claim shows that either: one of the d_ns is a source for y, or: we have a nested sequence of intervals whose length goes to zero. By the nested intervals lemma $\bigcap_{i=0}^{\infty}[a_n, b_n] = \{c\}$ with $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$. Since $c \in [a_0, b_0] = [a, b]$ then f is continuous at c.

In particular, $\lim_{n\to\infty} f(a_n) = f(c)$ and $\lim_{n\to\infty} f(b_n) = f(c)$. By item [i] $f(a_n) < y$ for all n therefore $\lim_{n\to\infty} f(a_n) \le y$ so $f(c) \le y$. By item [ii] $y < f(b_n)$ for all n therefore $y < \lim_{n\to\infty} f(b_n)$ so $y \le f(c)$. But $f(c) \le y$ and $y \le f(c)$ implies $y = f(c)$ thus we've found a source for y. $\square$