MATH 3210 - SUMMER 2008 - MIDTERM

You have an hour and a half to complete this test. Show all your work. The maximum grade is 100.

question	**grade**	**out of**
1		33
2		33
3a		16
3b		8
3c		6
3d		4
total		100

Student Number: ____________________

(1) (33 pts) Using the definition of a convergent sequence prove the following theorem (Do not appeal to any theorems):

If $\{a_n\}_{n=1}^{\infty}$ converges to a and $\{b_n\}_{n=1}^{\infty}$ converges to b then the sequence $\{a_n+2b_n\}_{n=1}^{\infty}$ converges to $a+2b$

Proof.

NTS: $\forall \varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$: $|a_n + 2b_n - (a+2b)| < \varepsilon$

Assumptions: $\forall \varepsilon' > 0$ there is an $N'(\varepsilon') \in \mathbb{N}$ such that for all $n > N'(\varepsilon')$: $|a_n - a| < \varepsilon'$

$\forall \varepsilon'' > 0$ there is an $N''(\varepsilon'') \in \mathbb{N}$ such that for all $n > N''(\varepsilon'')$: $|b_n - b| < \varepsilon''$

Calc:

$$|a_n + 2b_n - (a+2b)| = |(a - a_n) + (2b_n - 2b)| \leq^1$$
$$|a_n - a| + |2b_n - 2b| \leq^2 \varepsilon' + 2\varepsilon'' =^3 \varepsilon$$

(a) Inequality 1 follows from the triangle inequality.
(b) Inequality 2 holds for $n > N'(\varepsilon')$ and $n > N''(\varepsilon'')$.
(c) Equality 3 holds if $\varepsilon' = \frac{\varepsilon}{2}$ and $\varepsilon'' = \frac{\varepsilon}{4}$

Proof: Given $\varepsilon > 0$ take $\varepsilon' = \frac{\varepsilon}{2}$ to get $N_1 = N'(\frac{\varepsilon}{2})$ and $\varepsilon'' = \frac{\varepsilon}{4}$ to get $N_2 = N''(\frac{\varepsilon}{4})$.

Define $N(\varepsilon) = \max\{N_1, N_2\}$

If $n > N(\varepsilon)$ then by the calculation above: $|a_n + 2b_n - (a+2b)| < \varepsilon$ $\square$

(2) (33 pts) Consider the following sequence defined inductively:

$$a_1 = 1$$
$$a_{n+1} = \sqrt{4a_n + 1}$$

Prove that $\{a_n\}_{n=1}^{\infty}$ converges and find its limit.

Proof. We will prove that $\{a_n\}_{n=1}^{\infty}$ is monotonically increasing and bounded above. We then appeal to the monotone convergence theorem which says that:

Every sequence which is monotonic and bounded converges.

Therefore $\{a_n\}$ converges to some finite limit which we denote L.

We first compute L (which will help us choose an upper bound for $\{a_n\}$). Since $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$ it converges to L as well. From the main limit theorem we get that $\lim\limits_{n\to\infty} \sqrt{4a_n + 1} = \sqrt{4L+1}$. Therefore $a_{n+1} = \sqrt{4a_n+1}$ implies

$$\begin{aligned} &L = \sqrt{4L+1} && \Rightarrow \\ &L^2 = 4L + 1 && \Rightarrow \\ &L^2 - 4L - 1 = 0 \end{aligned}$$

The solutions to the above equation are $x_{1,2} = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$. Since $\sqrt{5} > 2$, $2 - \sqrt{5} < 0$ and L cannot be negative since we will show that a_n is monotonically increasing thus $a_n \geq a_1 = 1$. Therefore once we show that $\{a_n\}$ converges, its limit $L = 2 + \sqrt{5}$. Notice that $L < 2 + 3 = 5$.

Claim. $a_n < 5$ *for all* $n \in \mathbb{N}$

Proof of claim. We prove this by induction.

- Basis: We check this for $n = 1$: $a_1 = 1 < 5$
- Induction Hypothesis: $a_n < 5$
- Induction Step: $a_{n+1} <^{?} 5$

  $a_{n+1} = \sqrt{4a_n + 1}$. By the induction hypothesis $a_n < 5$ implies $\sqrt{4a_n+1} < \sqrt{4 \cdot 5 + 1} = \sqrt{21} < \sqrt{25} = 5$. Therefore $a_{n+1} < 5$

$\square$

Claim. *For all $n \in \mathbb{N}$: $a_{n+1} \geq a_n$*

Proof of Claim. We prove this by induction.

- Basis: We check this for $n = 1$: $a_2 \geq^? a_1$

  $a_1 = 1$, $a_2 = \sqrt{5}$ and $5 > 1$ implies $\sqrt{5} > \sqrt{1} = 1$
- Induction Hypothesis: $a_{n+1} \geq a_n$
- Induction Step: $a_{n+2} \geq^? a_{n+1}$

$$a_{n+2} = \sqrt{4a_{n+1} + 1}$$
$$a_{n+1} = \sqrt{4a_n + 1}$$

  By the induction hypothesis $a_{n+1} > a_n$ implies $4a_{n+1} + 1 > 4a_n + 1$ which implies $\sqrt{4a_{n+1} + 1} > \sqrt{4a_n + 1}$ hence $a_{n+2} > a_{n+1}$

$\square$

By the monotone convergence theorem $\{a_n\}$ converges and its limit is $2 + \sqrt{5}$

$\square$

(3) (34 pts) For each of the following statements, determine if they are true or false. If they are true, prove them. You are allowed and encouraged to appeal to the theorems proven in class (without proof) as long as you quote them in full. If the statement is false find a counter example.

(a) (16 pts) True/False:

If the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{a_n - b_n\}_{n=1}^{\infty}$ converge then $\{b_n\}_{n=1}^{\infty}$ converges.

True

Proof. If $\{a_n\}_{n=1}^{\infty}$, $\{a_n - b_n\}_{n=1}^{\infty}$ converge then by the main limit theorem, so does: $\{-(a_n - b_n) + a_n\}_{n=1}^{\infty}$. But $b_n = -(a_n - b_n) + a_n$ so b_n converges. $\square$

(b) (8 pts) True/False:

Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are sequences which satisfy the following properties:

(i) $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$

(ii) $b_n \neq 0$ for all $n \in \mathbb{N}$

then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$

False.

Counter Example: Take $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$ then $\lim\limits_{n\to\infty} a_n = 0$ (we proved this in class) and $\lim\limits_{n\to\infty} b_n = -\lim\limits_{n\to\infty} a_n = 0$ (by the main limit theorem. Moreover $b_n \neq 0$ for all n so these sequences satisfy all of the assumptions. However,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{-\frac{1}{n}} = -1$$

so the conclusion doesn't hold.

(c) (6 pts) True/False:

The sequence $a_n = (1 + \frac{1}{2^n+n})^{2^n+n}$ converges.

True.

Proof. We showed in class that the sequence $c_n = (1+\frac{1}{n})^n$ converges (by showing it was monotonically increasing and bounded above by 3). a_n is a subsequence of this sequence. Indeed, if $n_k = 2^k + k$ then $c_{n_k} = (1 + \frac{1}{2^k+k})^{2^k+k}$ is exactly a_n. By the theorem:

If a sequence c_k converges to L then every subsequence b_{n_k} converges to L

We get that the sequence a_n converges. $\square$

(d) (4 pts) True/False:

Consider the sequence $a_n = \cos(n)$ then:

There are natural numbers $m, l > 23$, $m \neq l$ such that:

$$|\cos(m) - \cos(l)| < \frac{1}{1000}$$

True

Proof. a_n is bounded. Indeed $|a_n| = |\cos(n)| \leq 1$

Bolzno-Weierstrauss Theorem: For any bounded sequence a_n there is a convergent subsequence a_{n_k}.

Since a_{n_k} converges, it has a finite limit L.

Thus: For all $\varepsilon > 0$ there is a $K(\varepsilon)$ such that for all $k > K(\varepsilon)$: $|a_{n_k} - L| < \varepsilon$

Taking $\varepsilon = \frac{1}{2000}$ there is a $K_1 = K(\frac{1}{2000})$ such that for all $k > K_1$: $|a_{n_k} - L| < \frac{1}{2000}$

Take $k > \max\{K_1, 23\}$ and $s = k + 1$. We calculate:[1]

$$|a_{n_k} - a_{n_s}| = |a_{n_k} - L + L - a_{n_s}| = |(a_{n_k} - L) - (a_{n_s} - L)|$$
$$\leq |a_{n_k} - L| + |a_{n_s} - L| < \tfrac{1}{2000} + \tfrac{1}{2000} = \tfrac{1}{1000}$$

Thus $|\cos(n_k) - \cos(n_s)| < \frac{1}{1000}$.

Lastly, we proved in class that $n_k \geq k$ (because n_k is strictly monotonically increasing).

Therefore $n_k \geq k > 23$ and $n_s \geq s > 23$ so we choose $m = n_k$ and $l = n_s$ and get:

$$|\cos(m) - \cos(l)| < \frac{1}{1000}$$

□

[1] A similar calculation actually shows that for all $k, s > K(\varepsilon)$: $|a_{n_k} - a_{n_s}| < 2\varepsilon$

In other words, from some place $K(\varepsilon)$ on, every two elements: a_{n_k}, a_{n_s} are 2ε close.