MATH 3210 - SUMMER 2008 - PRACTICE FINAL

You have two and a half hours to complete this test. Show all your work. There are a total of 105 points. The maximum grade is 100.

question	**grade**	**out of**
1		20
2a		15
2b		10
2c		10
3		15
4		10
5a		5
5b		5
5c		5
5d		10
total		105

Student Number: ____________________________

(1) (20 pts) State and prove the monotone convergence theorem. If you use other theorems in your proof you must state them in full but don't prove them.

Theorem. *If a_n is a monotonic sequence and there exists a constant K such that $|a_n| \leq K$ for all $n \in \mathbb{N}$ then a_n converges to a finite limit.*

Proof. Without loss of generality assume that a_n is monotonically increasing. Let $A = \{a_n | n \in \mathbb{N}\}$. A is bounded above by M. We use:

The completeness axiom: If A is bounded above then A has a superimum.

Let $s = \sup A$. We'll show:

$\forall \varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$: $|a_n - s| < \varepsilon$

Given $\varepsilon > 0$, since s is the superimum, $s - \varepsilon$ is not an upper bound for A. So, there exists an index N such that $s - \varepsilon < a_N$. Since a_n is monotonically increasing, for $n > N$: $a_n \geq a_N$. Therefore

$$s - \varepsilon < a_n < s + \varepsilon$$

Thus $|a_n - s| < \varepsilon$ for all $n > N$. □

(2) (35 pts)

(a) (15 pts) Prove by Cauchy's definition that $\lim_{x\to 2} \dfrac{x-3}{x+1} = -\dfrac{1}{3}$

Solution. NTS: For all $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if x satisfies $0 < |x-2| < \delta(\varepsilon)$ then $|\frac{x-3}{x+1} - (-\frac{1}{3})| < \varepsilon$

Calc: $|\frac{x-3}{x+1} - (-\frac{1}{3})| = |\frac{3(x-3)+(x+1)}{3(x+1)}| = |\frac{4x-8}{3(x+1)}| = \frac{4}{3}|\frac{x-2}{x+1}| = \frac{4}{3}\frac{|x-2|}{|x+1|}$.

If $\delta \leq 1$ then $|x-2| < 1$ and $1 < x < 3$ therefore $2 < x+1 < 4$ and $|x+1| > 2$ so $\frac{1}{|x+1|} < \frac{1}{2}$. Hence:

$$\left|\frac{x-3}{x+1} - \left(-\frac{1}{3}\right)\right| = \frac{4}{3}\frac{|x-2|}{|x+1|} < \frac{4}{3}\frac{|x-2|}{2} < \frac{2}{3}\delta < \delta \leq \varepsilon$$

The last inequality follows if $\delta \leq \varepsilon$.

Proof: Given $\varepsilon > 0$ let $\delta(\varepsilon) = \min\{1, \varepsilon\}$ and so for x such that $0 < |x-2| < \delta(\varepsilon)$ by the calculation above: $|\frac{x-3}{x+1} - (-\frac{1}{3})| < \varepsilon$ $\square$

(b) (10 pts) State the sequential characterization of $\lim_{x \to a} f(x) = L$ for a, L finite (this is Heine's definition).

Definition. $\lim_{x \to a} f(x) = L \iff$ *For all sequences* a_n *such that:*

(i) $a_n \neq a$ *for* $n \in \mathbb{N}$

(ii) $\lim_{n \to \infty} a_n = a$

it follows that $\lim_{n \to \infty} f(a_n) = L$.

(c) (10 pts) Consider $D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$ Prove that for any $a \in \mathbb{R}$ the limit $\lim_{x \to a} D(x)$ doesn't exist.

Proof. Take any $a \in \mathbb{R}$.

Claim.

- *There exists a sequence of* <u>*rational*</u> *numbers* $\{q_n\}_{n=1}^{\infty}$ *such that* $q_n \neq a$ *for all* n *and* $\lim_{n\to\infty} q_n = a$
- *There exists a sequence of* <u>*irrational*</u> *numbers* $\{r_n\}_{n=1}^{\infty}$ *such that* $r_n \neq a$ *for all* n *and* $\lim_{n\to\infty} r_n = a$

Proof of claim. By a consequence of the archimedean property, we know that between a and $a + \frac{1}{n}$ there is a rational number which we denote q_n and an irrational number which we denote r_n. Thus

$$a < q_n < a + \tfrac{1}{n} \quad \forall n \in \mathbb{N}$$
$$a < r_n < a + \tfrac{1}{n} \quad \forall n \in \mathbb{N}$$

Since both sides of both inequalities converge to a, applying the sandwich theorem we get $\lim_{n\to\infty} q_n = a$ and $\lim_{n\to\infty} r_n = a$ $\square$

By the definition of $D(x)$: $D(q_n) = 1$ and $D(r_n) = -1$ thus $\lim_{n\to\infty} D(q_n) = 1$ and $\lim_{n\to\infty} D(r_n) = -1$. By the sequential criterion for convergence, $\lim_{x\to a} f(x)$ doesn't exist. $\square$

(3) (15 pts) Compute the limit $\lim_{x\to 1} \dfrac{\int_1^{x^2} e^{\cos(t)}dt}{x-1}$. You must explain every step, and quote the theorems that you are using.

Proof. Remark: Since $\int_1^{1^2} e^{\cos(t)}dt = 0$ *this is a limit of the form* $\frac{0}{0}$. *We will argue that we can use L'Hopital's rule to compute it.*

- **The function** $F(x) = \int_1^x e^{\cos(t)}dt$ **is defined in a neighborhood of** 1.

  Denote $f(x) = e^{\cos(x)}$. Let $F(x) = \int_1^x f(t)dt$. This integral makes sense because: e^x is continuous everywhere, and $\cos(x)$ is continuous everywhere. Therefore, their composition $f(x) = e^{\cos(x)}$ is continuous everywhere, in particular it is continuous in the closed interval $[0,2]$. *A continuous function on a closed interval is integrable* and so $F(x)$ is defined in $[0,2]$ a closed neighborhood of 1.

- $F(x)$ **is differentiable at** $[0,2]$**.**

  By the Fundamental theorem of calculus 2, $F(x)$ is differentiable everywhere and $F'(x) = f(x)$ for all $x \in \mathbb{R}$.

- **Let** $G(x) = F(x^2)$**.** $G(x)$ **is differentiable on** $[0,2]$.

  Since x^2 is differentiable everywhere and $F(x)$ is differentiable everywhere, so is G (*a composition of differentiable functions is differentiable*) and by the chain rule

  $$G'(x) = 2x \cdot F'(x^2) = 2x \cdot f(x^2) = 2x \cdot e^{\cos(x^2)}$$

- **By the formula above** $G'(x)$ **is continuous at** 1 and $G'(1) = 2e^{\cos(1)}$.

- **Applying L'Hopital**.

  Define $h(x) = x - 1$, then $h'(x) = 1$. Since

  (a) $h(x) = x - 1 \neq 0$ for $x \in [0,2] \setminus \{1\}$ and $h'(x) \neq 0$ for all x.

  (b) $\lim_{x\to 1} G(x) = G(1) = 0$, and $\lim_{x\to 1} h(x) = h(1) = 0$ ($G(x), h(x)$ are continuous).

  (c) $\lim_{x\to 1} \dfrac{G'(x)}{h'(x)} = \dfrac{\lim_{x\to 1} G'(x)}{\lim_{x\to 1} h'(x)} = \dfrac{G'(1)}{h'(1)} = \dfrac{2e^{\cos(1)}}{1}$ (Here we're using arithmetics of limits and the fact that $G'(x)$ and $h'(x)$ are continuous at 1).

  then by L'Hopital's theorem: $\lim_{x\to 1} \dfrac{G(x)}{h(x)} = 2e^{\cos(1)}$

$\square$

(4) (10 pts) Prove: Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose $f(a) = g(a)$ and for all $x > a$: $f'(x) \leq g'(x)$ then for all $x > a$: $f(x) \leq g(x)$.

You must explain every step and quote the theorems that you're using.

Proof. Let $h(x) = g(x) - h(x)$. Then $h(x)$ is differentiable by arithmetics of differentiable functions. In addition $h(a) = g(a) - f(a) \geq 0$ and for all $x > a$: $h'(x) = g'(x) - f'(x) \geq 0$. By a consequence of the mean value theorem:

If $h : [a, b] \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and for all $x \in [a, b]$: $h'(x) \geq 0$ then $h(x)$ is monotonically increasing on $[a, b]$

We apply the theorem to h on $[a, x]$ then $h(x) \geq h(a) \geq 0$ thus $g(x) - f(x) \geq 0$ for $x > a$ and $g(x) \geq f(x)$. □

(5) (25 pts) For each of the following statements, determine if it is true or false. If the statement is false find a counter example. If it is true, prove it. You are allowed and encouraged to appeal to the theorems proven in class and in your homework as long as you quote them in full.

(a) (5 pts) True/False:

The equation $e^{-x} - e^{x} = \pi$ has a real solution.

True.

Solution. Define $f(x) = e^{-x} - e^{x} - \pi$. f is continuous since it is a combination of continuous functions. We know that $\lim_{x \to \infty} e^{x} = \infty$ and $\lim_{x \to -\infty} e^{x} = 0$. Thus, by arithmetics of infinite limits:

$$\lim_{x \to \infty} f(x) = 0 - \infty - \pi = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty - 0 - \pi = \infty$$

We have proven in a homework problem that if:

f is continuous, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$ then there exists a point c such that $f(c) = 0$.

Applying this to f we get a c such that $e^{-c} - e^{c} - \pi = 0$ $\square$

(b) (5 pts) True/False:

Every integrable function on $[a, b]$ is continuous on $[a, b]$

False.

Counter Example. Consider the function $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 2 & 1 < x \leq 2 \end{cases}$ This function is monotonically increasing on $[0, 1]$, we've proven:

If f is monotonically increasing on the interval $[a, b]$ then f is integrable on $[a, b]$

Therefore, our $f(x)$ it is integrable. However, it is not continuous at $x = 1$. It has a jump discontinuity since $\lim_{x \to 1+} f(x) = 2$ and $\lim_{x \to 1-} f(x) = 1$ $\square$

(c) (5 pts) True/False:

Every differentiable function on $[a, b]$ is continuous on $[a, b]$

True.

Proof. f is differentiable at x for every $x \in [a, b]$. We have proven in class that:

If f is differentiable at x then it is continuous at x.

Therefore, for all $x \in [a, b]$, f is continuous at x. $\square$

(d) (10 pts) True/False:

Suppose $f(x)$ is a function which satisfies:

$$|f(y) - f(x)| \leq K|y - x|^2$$

for some constant $K > 0$ and for all $x, y \in \mathbb{R}$.

Then $f'(x) = 0$ for all $x \in \mathbb{R}$.

True.

Proof. $0 \leq \left|\frac{f(y)-f(x)}{y-x}\right| = \frac{|f(y)-f(x)|}{|y-x|} < K|y - x|$ for all $x, y \in \mathbb{R}$. We use the sandwich theorem which says:

If k, g, h are functions such that for all y in a neighborhood of a, $k(y) \leq g(y) \leq h(y)$ and $\lim\limits_{y \to a} k(y) = \lim\limits_{y \to a} h(y) = L$ then $\lim\limits_{y \to a} g(y) = L$

We apply this for $k(y) = 0$, $g(y) = \frac{|f(y)-f(x)|}{|y-x|}$, $h(y) = K|y - x|$ and $a = x$. Then we get:

$$\lim_{y \to x} \left|\frac{f(y) - f(x)}{y - x}\right| = 0$$

But

$$-\left|\frac{f(y) - f(x)}{y - x}\right| \leq \frac{f(y) - f(x)}{y - x} \leq \left|\frac{f(y) - f(x)}{y - x}\right|$$

Hence, appealing to the sandwich theorem again,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$$

□