MATH 3210 - SUMMER 2008 - ASSIGNMENT #12

Integration

(1) Let $f(x) = c$ for all $x \in [a, b]$. Prove that for any partition P: $L(f, P) = c(b-a)$ and $U(f, P) = c(b-a)$. Conclude that f is integrable on $[a, b]$. What is the integral?

(2) (a) Let $h(x) \geq 0$ for all $x \in [a, b]$. Prove that $L(h, P) \geq 0$ for any partition P. Conclude that if h is integrable then $\int_a^b h(x)dx \geq 0$

(b) In addition to the hypothesis in 2a also assume that there is some point $c \in (a, b)$ where h is continous at c and $h(c) > 0$. Prove that $\int_a^b h(x)dx > 0$

(c) Consider the function $h(x) = \begin{cases} 0 & \text{if } x \neq 0, \frac{1}{2}, 1 \\ 1 & \text{if } x = 0, \frac{1}{2}, 1 \end{cases}$ prove that $\int_0^1 h(x) = 0$

(d) Suppose $f(x)$ and $g(x)$ are integrable on $[a, b]$ and for each $x \in [a, b]$: $f(x) \leq g(x)$. Prove that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Hint: you'll need to quote the theorem about the linearity of the integral here.

(3) Let f be an integrable function on $[a, b]$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$.

(a) Use exercises 1, 2 to show that

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

(b) Now assume that in addition, f is continuous on $[a, b]$. Prove that there exists a point $c \in [a, b]$ such that

$$f(c) = \int_a^b f(x)dx$$

Hint: You don't need the FTC here. This is much more elementary. It is called the theorem of the mean for integrals, since the term on the right looks a lot like the arithmatic mean of f on $[a, b]$.

(4) For each function, find its antiderivative and use the FTC1 to claculate the integrals

(a) $f(x) = x^3 + 4x^2 - 10x + 5$, $\int_0^1 f(x)dx =$

(b) $g(x) = \sin(x)$, $\int_0^{2\pi} g(x)dx =$

(c) $f(x) = \frac{1}{x}$, $\int_1^b f(x)dx =$

(5) Use the FTC2 to compute the following:

(a) $F(x) = \int_0^{x^2} \sin(t)dt$, $F'(x) =$

(b) $G(x) = \int_0^{e^x} \frac{t}{t+10} \ln(4t^2+5)dt$, $G'(x) =$

(c) Compute the limit: $\lim_{x \to 0} \dfrac{\int_0^{x^2} sin(t)dt}{x^3}$

(6) (bonus)

(a) Suppose $|f(x)| \leq M$ for all $x \in [a,b]$ and that for all $a < y < b$: f is integrable on $[y,b]$. Prove that f is integrable on $[a,b]$.

(b) Show that $f(x) = \frac{1}{x}$ is integrable for every interval $[y,1]$ when $0 < y < 1$ but not on $[0,1]$.

(c) Use the first part of this problem to show that the function $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

is integrable on $[-1,1]$.