MATH 3210 - SUMMER 2008

We wish to prove the formula:

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

Here's the rigorous formulation of this theorem:

Theorem. *If the sequence a_n converges to a and the sequence b_n converges to b then the sequence $c_n = a_n \cdot b_n$ also converges and its limit is $a \cdot b$*

Proof.

NTS: $\forall\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|a_n b_n - ab| < \varepsilon$ for every $n > N(\varepsilon)$.

Assumptions: (1) $\forall\varepsilon_a > 0$ there exist $N_a(\varepsilon_a) \in \mathbb{N}$ such that for all $n > N_a$: $|a_n - a| < \varepsilon_a$

(2) $\forall\varepsilon_b > 0$ there exist $N_b(\varepsilon_b) \in \mathbb{N}$ such that for all $n > N_b(\varepsilon_b)$: $|b_n - b| < \varepsilon_b$.

Calculation: $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| = |a_n(b_n - b) + (a_n - a)b| \leq^1 |a_n(b_n - b)| + |(a_n - a)b| = |a_n||b_n - b| + |a_n - a||b|$

(1) Inequality 1 follows from the triangle inequality.

(2) If we let $\varepsilon_a = \frac{\varepsilon}{2|b|}$ in the first assumption, then we get $N_a(\frac{\varepsilon}{2|b|})$. Let $N_1 = N_a(\frac{\varepsilon}{2|b|})$ then for every $n > N_1$ we get $|a_n - a| < \frac{\varepsilon}{2|b|}$.

Hence for $n > N_1$: $|a_n - a||b| < \frac{\varepsilon}{2|b|}|b| = \frac{\varepsilon}{2}$

(3) Let $\varepsilon_b = 1$ in the second assumption let $N_2 = N_a(\varepsilon_b)$ then for every $n > N_2$ we $|a_n - a| < 1$ which implies $|a_n| < |a| + 1$ (this step is left as an exercise to the reader - that means YOU!!).

Therefor for every $n > \max\{N_1, N_2\}$ we have:

$$|a_n b_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| < (|a| + 1)|b_n - b| + \frac{\varepsilon}{2}$$

(4) Now, let $\varepsilon_a = \frac{\varepsilon}{2(|a|+1)}$ (in assumption 1) and let $N_3 = N_b(\frac{\varepsilon}{2(|a|+1)})$ then for every $n > N_3$ we have $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$. Combine this with the former result and you get that for every $n > max\{N_1, N_2, N_3\}$.

$|a_n b_n - ab| < (|a|+1)|b_n - b| + \frac{\varepsilon}{2} < (|a|+1)\frac{\varepsilon}{2(|a|+1)} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Proof: Given $\varepsilon > 0$

Take $\varepsilon_a = \frac{\varepsilon}{2|b|}$ to get $N_1 = N_a(\frac{\varepsilon}{2|b|})$,

Take $\varepsilon_a = 1$ to get $N_2 = N_a(1)$,

Take $\varepsilon_b = \frac{\varepsilon}{2(|a|+1)}$ to get $N_3 = N_a(\frac{\varepsilon}{2(|a|+1)})$

Finally, let $N(\varepsilon) = \max\{N_1, N_2, N_3\}$. According to the assumptions, we know that for every $n > N$:

(1) $|a_n - a| < \frac{\varepsilon}{2|b|}$

(2) $|a_n| < |a| + 1$

(3) $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$

And by the calculation above:

$$|a_n b_n - ab| \leq \ldots \leq |a_n||b_n - n| + |a_n - a||b| < (|a|+1)\frac{\varepsilon}{2(|a|+1)} + \frac{\varepsilon}{2|b|}|b| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $\lim\limits_{n\to\infty} a_n b_n = ab$