Mapping Class Groups
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Last time:

Theorem 1 (Conjugacy classification in $\mathcal{MCG}(T^2)$)**.** *Each conjugacy class of elements $m \in \mathcal{MCG}(T^2) \approx \mathrm{SL}(2,\mathbf{Z})$ falls into exactly one of the following cases:*

Case 1: *If $|\mathrm{Tr}(m)| < 2$ then m has finite order 3, 4, or 6 and:*

- **Case 1a:** *If $|\mathrm{Tr}(m)| = 0$ then m is conjugate to exactly one of $t = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or t^3*
- **Case 1b:** *If $|\mathrm{Tr}(m)| = 1$ then m is conjugate to exactly one of $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, r^2, r^4, or r^5*

Case 2: *If $|\mathrm{Tr}(m)| = 2$ then m is conjugate to exactly one of $\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, $n \in \mathbf{Z}$.*

Case 3: *If $|\mathrm{Tr}(m)| > 2$ then, letting ϵ be the sign of $\mathrm{Tr}(m)$, there exists a sequence of positive integers $(p_1, p_2, \ldots, p_{2k-1}, p_{2k})$ where $k \geq 1$ such that*

- *m is conjugate to ϵ times the product of the word*

$$\underbrace{m_L \cdot \ldots \cdot m_L}_{p_1} \cdot \underbrace{m_R \cdot \ldots \cdot m_R}_{p_2} \cdot \ldots \cdot \underbrace{m_L \cdot \ldots \cdot m_L}_{p_{2k-1}} \cdot \underbrace{m_R \cdot \ldots \cdot m_R}_{p_{2k}}$$

where $m_L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $m_R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$

- *m is also conjugate to ϵ times the product of any cyclic permutation of this word,*
- *m is conjugate to no other positive or negative matrix.*

The sign ϵ and the sequence $(p_1, p_2, \ldots, p_{2k-1}, p_{2k})$, up to even cyclic permutation, is a complete conjugacy invariant of m.

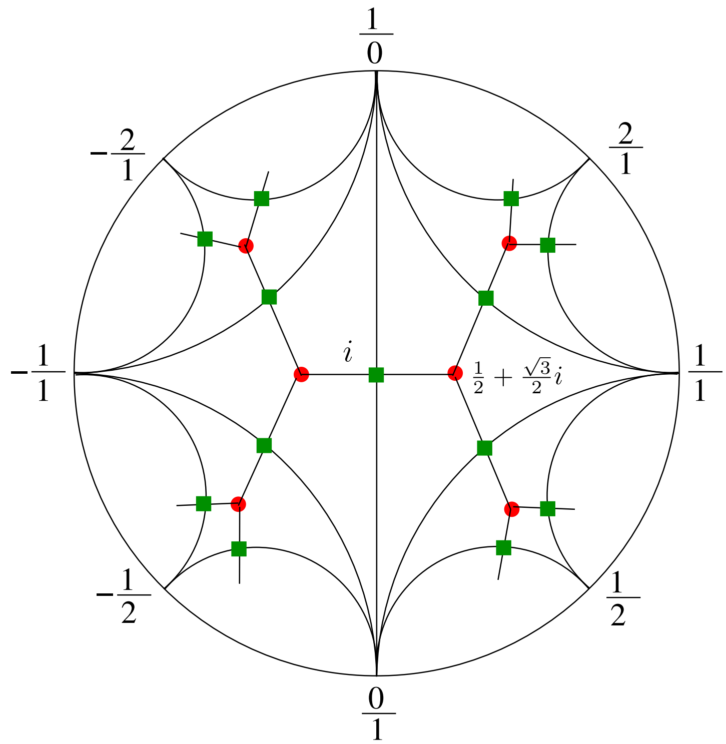


Figure 1: The dual tree to the Farey graph. A subdivided edge is a fundumental domain for the $\mathrm{PSL}(2,\mathbf{Z})$ action on the tree.

Proof of conjugacy classification in $\mathcal{MCG}(T^2)$ Given $m \in \mathcal{MCG}(T^2)$,

- Every nontrivial automorphism of a tree either fixes a vertex or translates along an invariant axis.
- Action of m on T is trivial if and only if $m = \pm\mathrm{Id}$.
- Suppose action of m is nontrivial.

  Case 1: m fixes a valence 2 or 3 vertex

  - but there is only one orbit of valence 2 vertices: the orbit of i. *(All points are specified in the upper half-plane model).*
  - and there is only one orbit of valence 3 vertices: the orbit of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$.
  - Conjugate m so that it fixes either:
    * the vertex i, or
    * the vertex $\frac{1}{2}+\frac{\sqrt{3}}{2}i$.

  Remark. *If A is an m-invariant set, then gA is invariant under gmg^{-1}. Suppose m fixes v. Since there are only two orbits of vertices, there's an element g such that $g(v) = i$ or $g(v) = \frac{1}{2}+\frac{\sqrt{3}}{2}i$. So gmg^{-1} fixes i or $\frac{1}{2}+\frac{\sqrt{3}}{2}i$.*

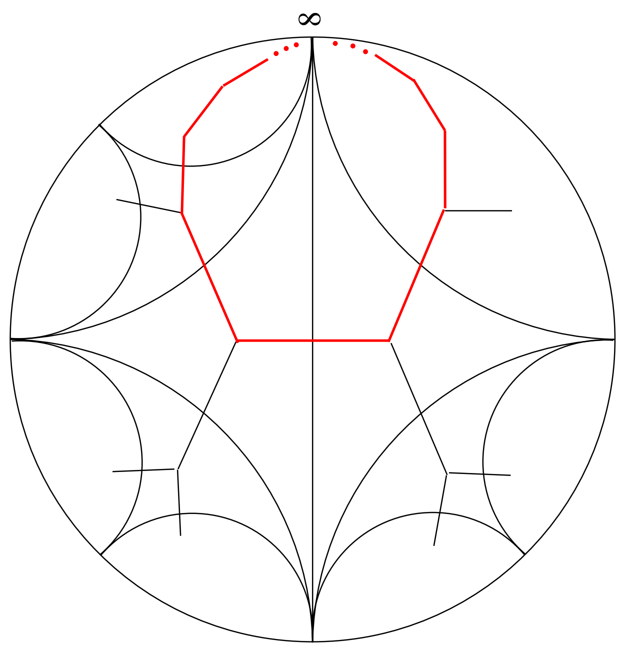


Figure 2: A horocyclic axis based at ∞

- Verify that:
  * only matrices that fix i are $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and its powers.
  * only matrices that fix $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ are $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and its powers.

Case 2: m translates along a horocyclic axis

- but there is only one orbit of horocyclic axes.
- Conjugate so that it translates along the horocyclic axis based at $\frac{1}{0} = \infty$

Remark. *If m's axis is based at $\frac{p}{q}$ then $g = \left(\begin{smallmatrix} 0 & 1 \\ q & p \end{smallmatrix}\right)$ will conjugate m to the desired element.*

- Verify matrices that translate along this axis are $\pm\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, where

$$n = \text{(oriented) translation distance}$$

Verification: If $m = \frac{az+b}{cz+d}$ and then ∞ will be taken to $\frac{a}{c}$. Since we want $\infty \to \infty$ then $c = 0$ and the determinant forces $a = d = \pm 1$.

Case 3: m translates along a nonhorocyclic axis, whose turn sequence has the form

$$L^{p_1}\, R^{p_2}\, \ldots\, L^{p_{2k-1}}\, R^{p_{2k}}$$

- But, there is only one orbit of nonhorocyclic axes for each turn sequence.

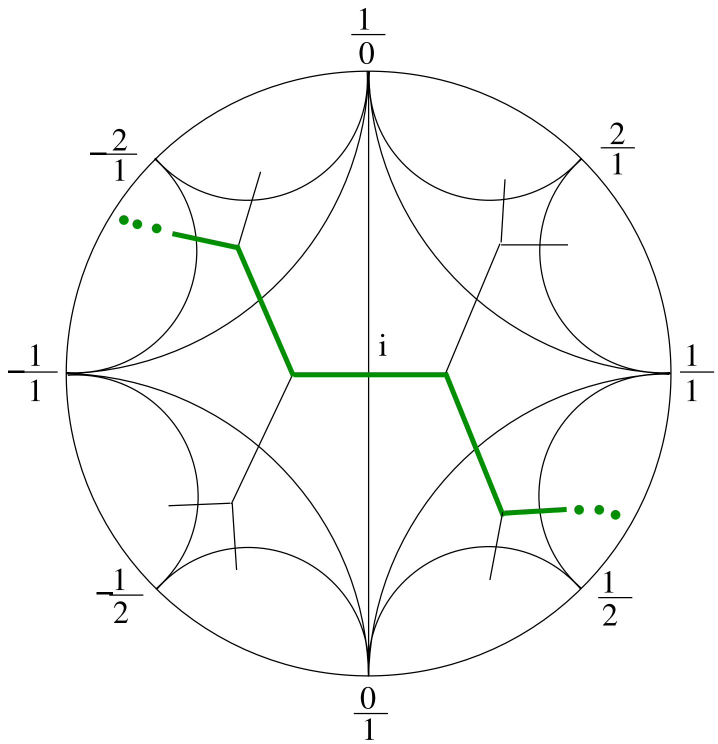


Figure 3: A non-horocyclic axis with turn sequence . . . RLRL . . .

– Conjugate to preserve axis passing through i from left to right.

Verification: let ℓ be the axis with given turn sequence (starting from a base edge and walking to the m-image of that base edge). Let ℓ_0 be the axis whose base edge passes through i from left to right with the same turn sequence. Choose $m_1 \in \mathrm{SL}(2,\mathbf{Z})$ to map the base edge of ℓ to the base edge of ℓ_0 (exists since there is only one orbit of edges). Since m' preserves the triangulation, the dual tree, and the cyclic orientation of edges of the dual tree, it takes ℓ to ℓ_0.

- Verify $m_L = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ takes $\overline{\frac{1}{0},\frac{0}{1}}$ to $\overline{\frac{1}{0},\frac{1}{1}}$
- Verify $m_R = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ takes $\overline{\frac{1}{0},\frac{0}{1}}$ to $\overline{\frac{1}{1},\frac{0}{1}}$
- Each time you move forward along another edge of the axis:
  - add R or L to the right of the turn sequence,
  - multiply m_R or m_L (resp.) on the right of the $\mathrm{SL}(2,\mathbf{Z})$ word.

Algorithm for conjugacy problem: In $\mathrm{PSL}(2,\mathbf{Z})$

- $\mathrm{PSL}(2,\mathbf{Z}) = \mathbf{Z}/2 * \mathbf{Z}/3 = \langle t \rangle * \langle r \rangle$
- Every word in t, r can be cyclically reduced to one of:
  - Case 1: Empty word
  - Case 2a: t
  - Case 2b: r or r^2
  - Case 3a: Power of tr or of tr^2
  - Case 3b: Word in tr and tr^2, with at least one of each.
- Example:

$$w = t^{17} r^{43} t^3 r^5 t^{-8} r^6 = trtr^2tr^2$$
$$tr \leftrightsquigarrow M_L \quad tr^2 \leftrightsquigarrow M_R$$
$$w \leftrightsquigarrow LR^2$$

Constructing an Anosov homeomorphism.

We wish to restate and reprove the Anosov portion of Theorem 1 using a more topological method using the torus (rather than the combinatorial method using the tree) that will generalize to all hyperbolic surfaces $\Sigma_{g,p}$.

The following theorem is useful in many areas of mathematics.

Theorem 2 (Perron-Frobenius Theorem)**.**
For each positive $n\times n$ integer matrix M there exists a unique positive unit eigenvector v, and it has eigenvalue $\lambda > 1$.

Sketch of the Proof : A positive matrix takes positive vectors to positive vectors, so $\{x \in \mathbf{R}^n | x_i > 0\}$ is invariant. Consider the simplex $S = \{x \in \mathbf{R}^n | x_i > 0, \Sigma x_i = 1\}$ and the map $f : \bar{S} \to \bar{S}$ given by applying M and normalizing. By Brower there's a fixed point, which gives us a non-negative eigen-vector v. For some j, $v_j \neq 0$, but then for each i: $\lambda v_i = M_i \cdot v \geq M_{i,j} \cdot v_j > 0$, hence $v_i > 0$ for all i. Since all entries are integer λ must be greater than 1. Uniquness follows from the fact that f is contracting. $\diamondsuit$

- Each $m \in \mathrm{SL}(2,\mathbf{Z})$ with $\mathrm{Tr}(m) > 2$ is conjugate to a positive matrix. (Any product as in Theorem 1 produces a positive matrix)
- By Perron-Frobenius, m has a unit eigenvector v^u with eigenvalue $\lambda > 1$.
- Since $\det(m) = 1$, there is also a unit eigenvector v^s with eigenvalue λ^{-1}.
- Let $\widetilde{\mathcal{F}}^u$ = foliation of $\mathbf{R}^2$ by lines parallel to v^u.

- $\widetilde{\mathcal{F}}^u$ is preserved by the linear action of m on $\mathbf{R}^2$.
- Each line of $\widetilde{\mathcal{F}}^u$ is metrically stretched by factor λ.

- Let $\widetilde{\mathcal{F}}^s$ = foliation of $\mathbf{R}^2$ by lines parallel to v^s,
  - $\widetilde{\mathcal{F}}^s$ is preserved by linear action of m,
  - Each line of $\widetilde{\mathcal{F}}^u$ is metrically compressed by factor λ.
- Action of $\mathbf{Z}^2$ preserves $\widetilde{\mathcal{F}}^u, \widetilde{\mathcal{F}}^s$ and the metrics along their leaves.
- Therefore, action of m on T^2 preserves projected foliations $\mathcal{F}^u, \mathcal{F}^s$, stretching leaves of $\mathcal{F}^u$ by λ, compressing leaves of $\mathcal{F}^s$ by λ.

Remark: This implies that no leaf of the unstable or stable foliations is closed. Indeed, if some leaf were closed, then it would have to be a simple closed curve (since it is a leaf of a foliation). But then $m(\alpha)$ would be a simple closed curve which is a multiple of α! This in turn, implies that the slope of the eigenvectors of m are irrational. Since m is an integer matrix λ is irrational as well, for if λ were rational then the linear system of equations whose solution gives the eigenvector would have rational coefficients and so the slope of the eigenvector would be rational.

- Action of m on T^2 is said to be *Anosov.*
- Some additional useful features:
  - Each leaf of $\mathcal{F}^u, \mathcal{F}^s$ is dense in T^2.
  - There is a Euclidean metric on T^2, affinely equivalent to the given metric, in which $\mathcal{F}^s, \mathcal{F}^u$ intersect at right angles, and locally m looks like $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Anosov conjugacy classification, method 2

- Let $\Phi\colon T^2 \to T^2$ be the Anosov homeomorphism representing $m \in \mathrm{SL}(2, \mathbf{Z})$ with $\mathrm{Tr}(m) > 1$.
- Stable and unstable foliations $\mathcal{F}^s, \mathcal{F}^u$.
- Expansion factor $\lambda > 1$.
- For each $r > 0$ let ℓ_r denote a $\mathcal{F}^u$ segment of length r
  - choice of ℓ_r is irrelevant, because the Lie group structure on T^2 acts transitively on such segments, preserving $\mathcal{F}^s, \mathcal{F}^u$.
- Goals:
  - define a *rectangle decomposition* and a *train track*, obtained by "slicing along ℓ_r".
  - use these to obtain anew the RL loop.

Rectangle decomposition

- There is a unique decomposition of T^2 into rectangles $\{R_i\}$, said to be obtained by "slicing along ℓ_r", as follows:
  - R_i is (image of) map $I \times I \to T^2$ which is an embedding on its interior and on the interior of each side.
  - Horizontals ($I \times$ point) go to segments of $\mathcal{F}^u$
  - Verticals (point $\times I$) go to segments of $\mathcal{F}^s$.
  - Each horizontal side of R_i is in ℓ_r
  - The interior of each rectangle is disjoint from ℓ_r.
  - R_i is maximal with respect to these properties.
- Maximality $\implies$ each vertical side of R_i contains an endpoint of ℓ_r, coinciding with either the top endpoint of the side, the bottom endpoint, or some interior point.

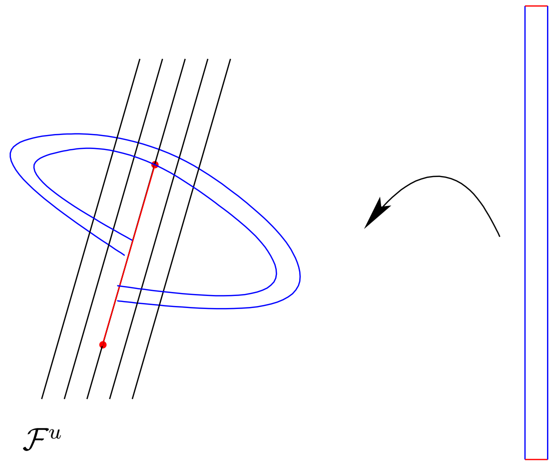


Figure 4: To construct a rectangle decomposition, start with a point p on ℓ_r and slide along the leaf of $\mathcal{F}^s$ until returning to ℓ_r. Since leaves are dense, and the foliations transverse to one another, we must eventually return to ℓ_r. Do the same to points in a small neighborhood of p along ℓ_r, to obtain an embedded rectangle, whose vertical lines are leaves of $\mathcal{F}^s$ and whose horizontal lines are leaves of $\mathcal{F}^u$. We can continue "beefing up" the rectangle, until one of it's vertices coincides with an endpoint of ℓ_r or one of its vertical edges collides with an endpoint of ℓ_r. After doing the same to all points on ℓ_r we get a maximal rectangle decoposition of the torus. To check that this covers the torus, for $p \in \mathcal{T}^2$ which is not on ℓ_r, there is a unique leaf of $\mathcal{F}^s$ which passes through p. Continuing along it in either direction, until reaching a point q of ℓ_r (we must stop by the density of the leaves), the points p and q belong to the same rectangle.

Train track construction

- Collapse each vertical segment of each R_i.
- Result: a *train track* τ_r, with *branches* and *switches*.
  - Split open along ℓ_r
  - Generic picture:
    - 3 rectangles, 2 vertical segments
    - τ_r has 3 branches and 2 switches
    - This is generic in that, if r is perturbed, the picture does not change *up to isotopy of* T^2.
  - Nongeneric picture:
    - 2 rectangles, 1 vertical segment
    - τ_r has 2 branches and 1 switch.
    - This is nongeneric in that:

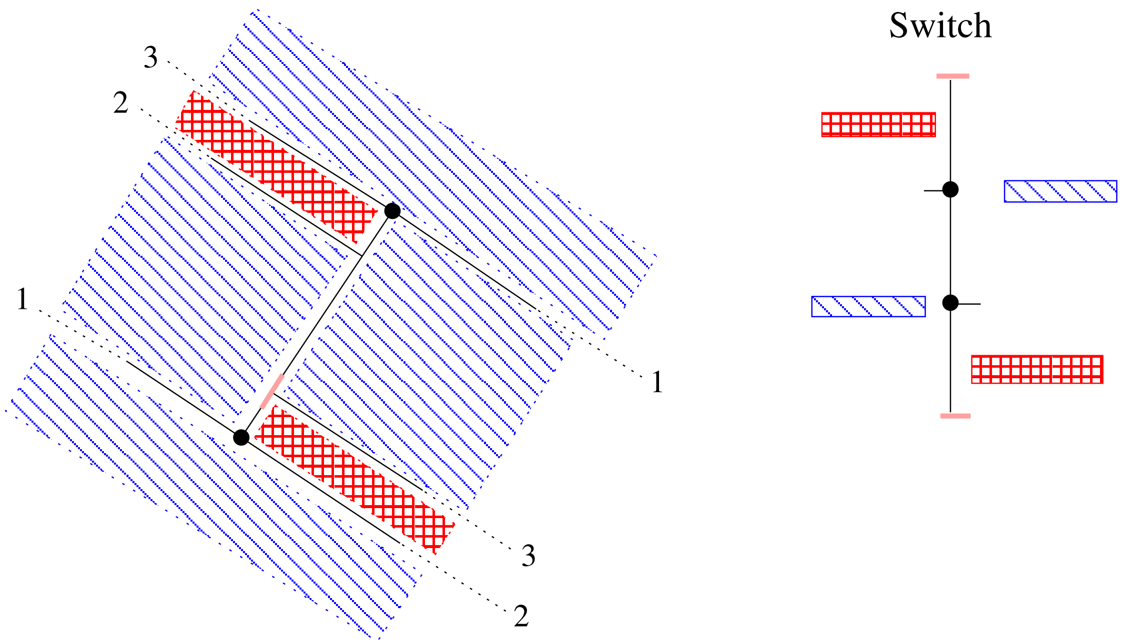


Figure 5: In the non-generic picture, we have one switch corresponding to moving the pink segment along the foliation. Here all branch segment meet at the same switch. A turn is legal if there is a leaf of the unstable folitation which makes that turn.

· if r is decreased a little the picture changes: the switch is combed out.

· if r is increased a little the picture changes to a particular generic picture, well defined up to isotopy: either a Left split or a Right split.

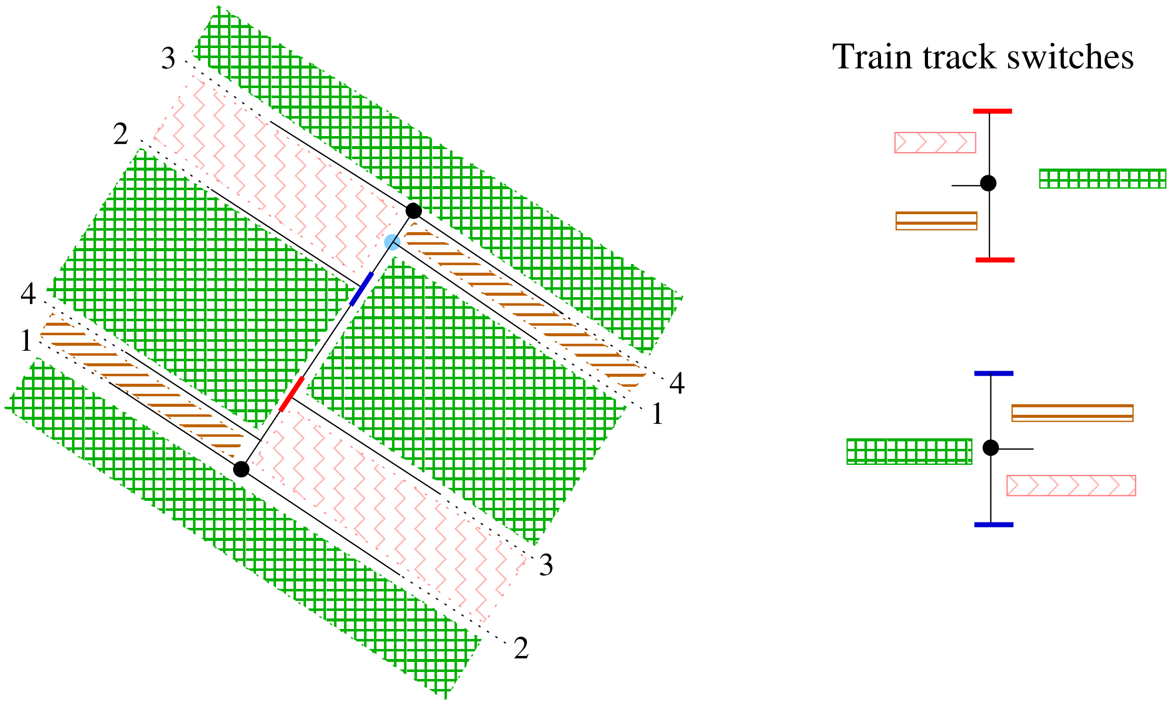
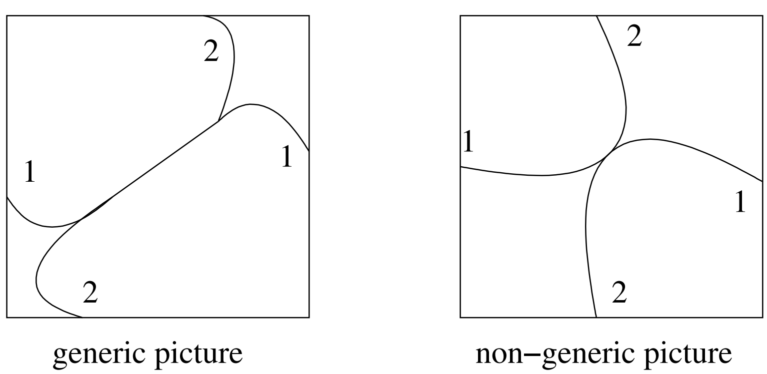


Figure 6: We get this generic picture from the non-generic case by moving one of the endpoints of the segment ℓ_r to increase r. The blue dot marks where the previous segment ended. In this picture we see 3 rectagles, 3 branches and two switches.



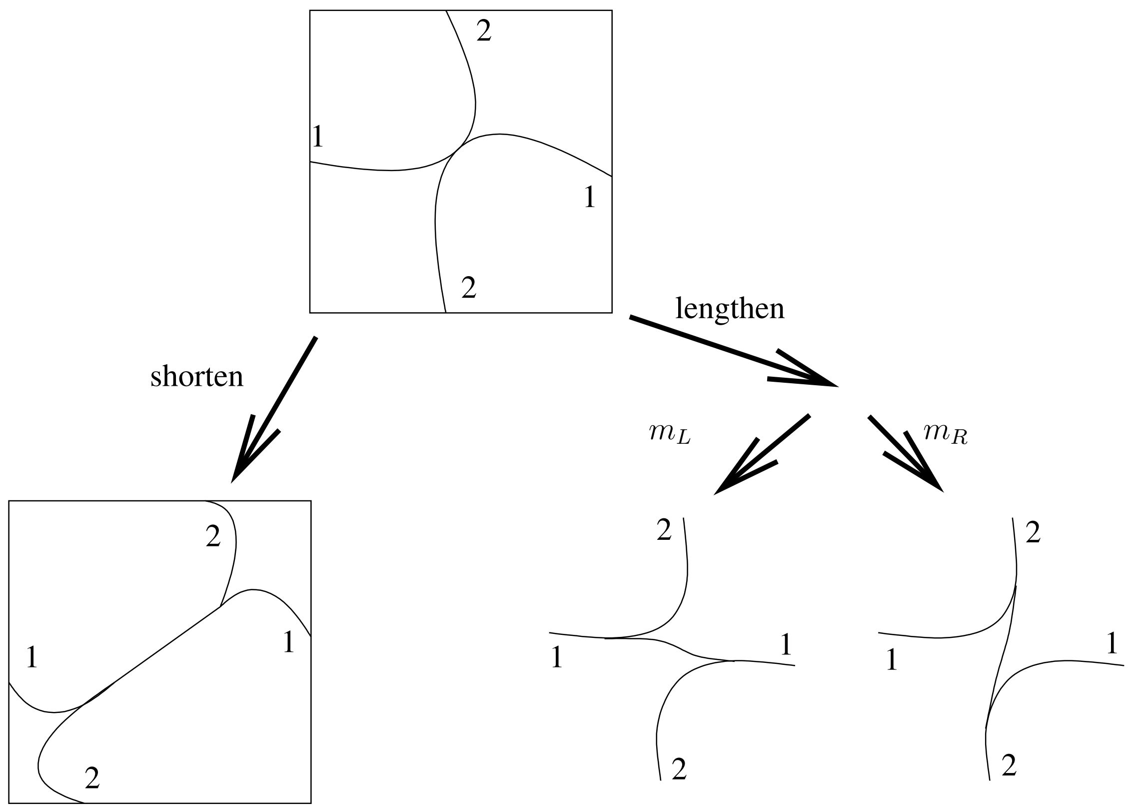


Figure 7: When we have a non-generic rectangle decomposition, and we perturb r, we get a generic picture. The kind of generic picture we get will determin the RL sequence of our homomorphism.

Train track expansion of Φ

- bi-infinite sequence of nongeneric points
- induced bi-infinite seuqence of L's and R's
- Φ acts, stretching ℓ_r to $\ell_{\lambda r}$.
- Induces loop of L's and R's
- This is the *same loop* as obtained using the action on the dual tree of the modular diagram.
- Proof: Left is m_L, right is m_R.