Homework 2: More on the hyperbolic plane

Let $a, b \in \overline{U}$, we denote the hyperbolic geodesic between them by $[a, b]$. For any set $X \subset U$ the r neighborhood of X is

$$N_r(X) = \{y \in U \mid \exists x \in X \text{ so that } d(x, y) < r\}$$

Definition 1. A geodesic triangle $\Delta(a, b, c)$ is δ-thin if $[a, b] \subseteq N_\delta([b, c] \cup [c, a])$, $[b, c] \subseteq N_\delta([a, b] \cup [a, c])$, $[a, c] \subseteq N_\delta([b, c] \cup [a, b])$.

1. Let $g_1 : \mathbb{R} \to U$ be $g_1(t) = ie^t$ and $g_2 : [0, \pi] \to U$ be $g_2(t) = e^{it}$. Describe $N_r(\mathrm{Im} g_1)$ and $N_r(\mathrm{Im} g_2)$.
2. What is the hyperbolic distance from i to $2i + 1$.
3. Show that there exists a constant $\delta > 0$ so that all hyperbolic triangles are δ thin. Find the minimal δ that works. (Hint: first show for ideal triangles).
4. Generalize Gauss-Bonnet's theorem to polygons with n sides. That is, if P is a convex polygon with $n \geq 3$ sides and with angles $\alpha_1, \ldots, \alpha_n$. Find a formula for the area depending only on the angles.
5. Prove that hyperbolic isometries preserve Euclidean angles.