Homework 4: The Classification of Isometries and incomplete hyperbolic surfaces.

Definition 1. Let (X, d) be a metric space and $f : X \to X$ an isometry. The translation length of f is

$$\tau(f) = \inf\{d(x, f(x)) \mid x \in X\}$$

Definition 2. Let (X, d) be a (non-positively curved) metric space, and $f \in \mathrm{Isom}(X)$. Then

- f is *elliptic* if f has a fixed point in X, i.e. $\exists x \in X$ so that $f(x) = x$.
- f is *hyperbolic* if its action has an axis: There is a geodesic line ℓ so that $f(\ell) = \ell$ and f acts by translations on ℓ that is if $\ell(t)$ is a parameterization of ℓ according to arc-length then $f(\ell(t)) = \ell(t+\tau(f))$.
- f is parabolic if $d(x, f(x)) > \tau(f)$ for all $x \in X$.

Now we consider $X = \mathbb{H}^2$ the hyperbolic plane. We use both the upper-half plane and the disc models.

Definition 3. A horocycle based at ∞ is a set of the form $\{z \mid \mathrm{Im}(z) = c\}$ for some c. For any point z_0 there is a unique horocycle based at ∞ going through z_0 and we denote it by $H_\infty(z_0)$. Notice that if ℓ is a geodesic with an endpoint at ∞ then ℓ is perpendicular to all horocycles based at ∞. A horocycle based at $r \in \mathbb{R}$ is a set of the form $gH_\infty(z_0)$ where $g \in Mob(\mathbb{R})$ satisfies $g(\infty) = r$. A horocycle is an isometric image of a horocycle based at ∞.

1. Show that if there are $x \neq y \in \mathbb{H}^2$ so that $f(x) = x$ and $f(y) = y$ then $f = \mathrm{id}$.
2. Show that if f is elliptic and x is its fixed point then the set $S(x, r) = \{z \mid d(x, z) = r\}$ is invariant under f.
3. Consider $f \neq \mathrm{id}$ as a map in $Mob(\mathbb{R})$. Show that f is conjugate to $g(z) = \frac{\cos t \cdot z + \sin t}{-\sin t \cdot z + \cos t}$ by some map $h \in Mob(\mathbb{R})$. Find $\mathrm{trace}(A_f)$ where A_f is the matrix in $SL_2(\mathbb{R})$ which corresponds to f.
4. Show that if f is an isometry so that $0 < \tau(f) = d(x, f(x))$ for some $x \in X$ then f is hyperbolic. Show that the axis is unique.

5. Show that if f is hyperbolic then it is conjugate to $g(z) = \lambda z$ for some $\lambda \in \mathbb{R}$. Find trace(A_f).

6. Show that if f is parabolic then there is a point $x \in \partial U$ so that $f(x) = x$. Prove that f is conjugate to $g(z) = z + c$ for some $c \in \mathbb{R}$. Find trace(A_f).

7. Consider the ideal quadrilateral bounded by the following geodesic lines:
$$\begin{aligned} l_1 &= \{\cos t + i \sin t \mid 0 < t < \pi\} \\ l_2 &= \{z \mid \mathrm{Re}\{z\} = 3\} \\ l_3 &= \{2 + \cos t + i \sin t \mid 0 < t < \pi\} \\ l_4 &= \{z \mid \mathrm{Re}\{z\} = -1\} \end{aligned}$$
Let $f(z) = \frac{4z+1}{z+1}$ and $g(z) = \frac{4z-6}{-z+3}$.

   (a) Draw a sketch of this quadrilateral, and show that f takes l_1 to l_2 and g takes l_3 to l_4. Find f^{-1} and g^{-1}. What is the topological space obtained by this gluing? Does the gluing induce a hyperbolic metric on this surface?

   (b) Find $f(i)$ and $g(2+i)$

   (c) Let $x_1 = 3 + 2i$. We construct a sequence $x_1, x_1', \ldots, x_4, x_4'$ as follows. Let $x_1 = 3 + \frac{3}{2}i \in l_2$ let $x_1' = f^{-1}(x_1) \in l_1$. Let x_2 be the unique point of $H_{-1}(x_1') \cap l_4$. $x_2' = g^{-1}(x_2')$ and $\{x_3\} = H_1(x_2') \cap l_1$. $x_3' = f(x_3)$ and $x_4 = H_3(x_4) \cap l_3$. Let $x_4' = g(x_4)$ and $x_0 = H_\infty(x_1) \cap l_4$. Notice that $x_0, x_4' \in l_4$. Prove that $\mathrm{Im} x_0 < \mathrm{Im} x_4'$.

   (d) Suppose y_1 was a different point in l_2 and construct a similar sequence $y_1, y_1', \ldots, y_4, y_4'$ and y_0. Prove that $d(x_0, x_4') = d(y_0, y_4')$. This number is the gluing invariant at ∞ denoted $d[\infty]$.

   (e) Suppose we continue this sequence $x_1, x_1', \ldots, x_n, x_n', \ldots$ prove that the sequence $\{x_{4m}\}_{m=1}^{\infty}$ descends to a Cauchy sequence on the glued surface which doesn't converge to any point.

   This construction shows that if the gluing invariant is not zero, then the glued hyperbolic surface is incomplete.