Homework 6: The Uniformization Theorem

For background, read pages 25-42 in Imayoshi-Taniguchi.

1. Consider $A = \{w \mid r < |w| < 1\}$ and let λ be defined by $r = \exp\left(-\frac{2\pi^2}{\log\lambda}\right)$. Let $\pi : U \to A$ be defined by $\pi(z) = \exp\left(2\pi i \frac{\log z}{\log \lambda}\right)$ where log is the principle branch of the logarithmic function. Prove that π is conformal covering map.

2. Prove the following theorem (you may consult Imayoshi and Taniguchi page 33):

   (a) If $\gamma \in \mathrm{Aut}\big(\hat{\mathbb{C}}\big)$ then γ has the form $\gamma(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

   (b) If $\gamma \in \mathrm{Aut}\big(\mathbb{C}\big)$ then γ has the form $\gamma(z) = az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$.

   (c) If $\gamma \in \mathrm{Aut}\big(\hat{\Delta}\big)$ then γ has the form $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ where $a, b \in \mathbb{C}$ and $|a| - |b| = 1$.
   It can also be written as $\gamma(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ where $\theta \in [0, 2\pi)$ and $\alpha \in \Delta$.

   (d) If $\gamma \in \mathrm{Aut}\big(U\big)$ then γ has the form $\gamma(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

3. Suppose that R is a Riemann Surface, and that $\widetilde{R} \cong \Delta$. Show that R gives rise to a complete hyperbolic structure R' on the same underlying surface.

4. Suppose $f : R \to S$ is a holomorphic map where R, S are Riemann Surfaces and $\widetilde{R}, \widetilde{S} \cong \Delta$. Show that f is a local isometry from R' to S'. Show that if f is biholomorphic then f is an isometry from R' to S'.

5. Let X be a complete hyperbolic surface. Show that X gives rise to a conformal structure X^* on the underlying surface. Prove that if $f : X \to Y$ is an isometry then $f : X^* \to Y^*$ is biholomorphic.